

# Recent Advances in Stochastic Gravity: Theory and Issues

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Whereas semiclassical gravity is based on the semiclassical Einstein equation with sources given by the expectation value of the stress-energy tensor of quantum fields, stochastic semiclassical gravity is based on the Einstein-Langevin equation, which has in addition sources due to the noise kernel. The noise kernel is the vacuum expectation value of the (operator-valued) stress-energy bi-tensor which describes the fluctuations of quantum matter fields in curved spacetimes. We show how a consistent stochastic semiclassical theory can be formulated as a perturbative generalization of semiclassical gravity which describes the back reaction of the lowest order stress-energy fluctuations. The original approach [1] used in the early investigations leading to the establishment of this field [2] (for a review emphasizing ideas, see [3]) was based on quantum open system concepts (where the metric field acts as the “system” of interest and the matter fields as part of its “environment”) and the influence functional method. Here, following Refs. [4,5] we first give an axiomatic derivation of the Einstein-Langevin equations and then show how they can also be derived by the original method based on the influence functional. As a first application we solve these equations following Ref. [6], and compute the two-point correlation functions for the linearized Einstein tensor and for the metric perturbations in a Minkowski background. We then turn to the important issue of the validity of semiclassical gravity by examining the criteria based on the ratio of the variance of fluctuations of the stress-energy tensor of a quantum field to its mean. We show a calculation of these quantities performed in Ref. [7,8] for a massless scalar field in the Minkowski and the Casimir vacua as a function of an intrinsic scale defined by introducing a smeared field or by point separation. Contrary to prior claims, the ratio of variance to mean-squared being of the order unity does not necessarily imply the failure of semiclassical gravity. Expressions for the variance to mean-squared ratio as a function of the ratio of the intrinsic to extrinsic scale (defined by the separation of the plates or the periodicity of space for the Casimir topology) identifying the spatial extent where negative energy density prevails are useful for studying quantum field effects in worm holes, baby universe and the design feasibility of ‘time-machines’. This study also raises new questions into the very meaning of regularization of the stress-energy tensor for quantum fields in curved or ordinary flat spacetimes. It corroborates the view held by one of us [3] that at high energies it is more useful to consider spacetime as possessing an extended structure, with the stress-energy bi-tensor of quantum matter fields as the source. To compare with the more familiar point-defined quantum field theory operative at low energies, one needs to address the issue of regularization of the noise kernel. This we discuss following the work of Ref. [9] which uses the point-separation method to derive a general expression for a regularized noise-kernel for quantum fields in general curved spacetimes. From these expressions describing the behavior of fluctuations of quantum fields one could investigate a host of important problems in early universe and black hole physics. This includes addressing the viability of a vacuum-dominated phase which inflationary cosmology is predicated upon, examining how structures are seeded, how the horizon behavior of a black hole is altered and how the late time dynamics of black holes is affected with backreaction of Hawking radiation. We intend to discuss these applications in a later report.

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## I. OVERVIEW

Stochastic semiclassical gravity (STG) is a theory developed in the Nineties using semiclassical gravity (SCG, quantum fields in classical spacetimes, solved self-consistently) as the starting point and aiming at a theory of quantum gravity (QG) as the goal. While semiclassical gravity is based on the semiclassical Einstein equation with the source given by the expectation value of the stress-energy tensor of quantum fields, stochastic gravity (we will often use the shortened term *stochastic gravity* as there is no confusion in the source of stochasticity in gravity being due to quantum fields here, and not of a classical origin) includes also its fluctuations in a new stochastic semiclassical or the Einstein-Langevin equation. If the centerpiece in semiclassical gravity theory is the vacuum expectation value of the stress-energy tensor of a quantum field, and the central issues being how well the vacuum is defined and how the divergences can be controlled by regularization and renormalization, the centerpiece in stochastic semiclassical gravity theory is the stress-energy bi-tensor and its expectation value known as the noise kernel. The mathematical properties of this quantity and its physical content in relation to the behavior of fluctuations of quantum fields in curved spacetimes are the central issues of this new theory. How they induce metric fluctuations and seed the structures of the universe, how they affect the black hole horizons and the backreaction of Hawking radiance in black hole dynamics, including implications on trans-Planckian physics, are new horizons to explore. On the theoretical issues, stochastic gravity is the necessary foundation to investigate the validity of semiclassical gravity and the viability of inflationary cosmology based on the appearance and sustenance of a vacuum energy-dominated phase. It is also a useful beachhead supported by well-established low energy (sub-Planckian) physics to explore the connection with high energy (Planckian) physics in the realm of quantum gravity.

In these lectures we want to summarize our work on the theory aspects since 1998. (A review of ideas leading to stochastic gravity and further developments originating from it can be found in Ref. [1,3]; a comprehensive formal description is given in Refs. [4,5]). It is in the nature of a progress report rather than a review. In fact we will have room only to discuss two or three theoretical topics of basic importance. We will try to mention all related work so the reader can at least trace out the parallel and sequential developments. The references at the end of each topic below are representative work where one can seek out further treatments.

Stochastic gravity theory is built on three pillars: general relativity (GR), quantum fields (QF) and nonequilibrium (NEq) statistical mechanics. The first two uphold semiclassical gravity, the last two span statistical field theory, or NEqQF. Strictly speaking one can understand a great deal without appealing to statistical mechanics, and we will try to do so here. But concepts such as quantum open systems [10] and techniques such as the influence functional [11] (which is related to the closed-time-path effective action [12,13]) were a great help in our understanding of the physical meaning of issues involved towards the construction of this new theory, foremost because quantum fluctuations and correlation have become the focus. Quantum statistical field theory and the statistical mechanics of quantum field theory [14–17] also aided us in searching for the connection with quantum gravity through the retrieval of correlations and coherence. We show the scope of stochastic gravity as follows:

### I. Ingredients

- A. From General Relativity [18,19] to Semiclassical Gravity.
- B. Quantum Field Theory in Curved Spacetimes [20–23]:
  - Stress-energy tensor: Regularization and renormalization.
  - Self-consistent solution: Backreaction problems [24].
  - Effective action: Closed time path, initial value formulation [12,13].
  - Equation of motion: Real and causal.
- C. Nonequilibrium Statistical Mechanics:
  - Open quantum systems [10].
  - Influence Functional: Stochastic equations [11].
  - Noise and Decoherence: Quantum to classical transition [25,26].
- D. Decoherence in Quantum Cosmology and Emergence of Classical Spacetimes [27]

### II. Theory

- A. Dissipation from Particle Creation [28,29].
  - Backreaction as Fluctuation-Dissipation Relation (FDR) [30].
- B. Noise from Fluctuations of Quantum Fields [1,32,31].
- \* C. Einstein-Langevin Equations [2,4,5].

\* D. Metric Fluctuations in Minkowski spacetime [6].

### III. Issues

\* A. Validity of Semiclassical Gravity [7,8].

B. Viability of Vacuum Dominance and Inflationary Cosmology.

\* C. Stress-Energy Bitensor and Noise Kernel: Regularization Reassessed [9].

### IV. Applications: Early Universe and Black Holes

[Test field treatment:]

A. Wave Propagation in Stochastic Geometry [33].

B. Black Hole Horizon Fluctuations: Spontaneous/Active versus Induced/Passive [34–38].

[Including backreaction:]

C. Noise induced inflation [39].

D. Structure Formation: [40], Trace anomaly-driven inflation [41–44].

E. Black Hole Backreaction as FDR [45–48].

### V. Related Topics

A. Metric Fluctuations and Trans-Planckian Problem [37,49,50].

B. Spacetime Foam [51].

C. Universal ‘Metric Conductance’ Fluctuations [52].

### VI. Ideas

A. General Relativity as Geometro-Hydrodynamics [53].

B. Semiclassical Gravity as Mesoscopic Physics [54].

C. From Stochastic to Quantum Gravity:

via Correlation hierarchy of interacting quantum fields [3,15,17].

Possible relation to string theory and matrix theory [55].

We list only the latest work in the respective topics above describing ongoing research. The reader should consult the references therein for earlier work and the background material. We do not seek a complete coverage here, but will discuss only the selected topics in theory and issues (Parts II and III above), i.e., those topics marked with \*s above, in Sections 2-5 below. In a later report we plan to discuss recent advances in the applications (Parts IV-V above). We use the (+, +, +) sign conventions of Refs. [18,19], and units in which  $c = \hbar = 1$ .

## II. STOCHASTIC SEMICLASSICAL GRAVITY

In this section we introduce stochastic semiclassical gravity. We will start with semiclassical gravity which assumes that the stress-energy fluctuations of the matter fields are negligible. When these fluctuations cannot be neglected we may perturbatively correct the semiclassical theory to take them into account. This will lead to the semiclassical Einstein-Langevin equations [2] as the dynamical equations which describe the back reaction of quantum stress-energy fluctuations on the gravitational field. We will introduce these equations by two independent methods. The first, based on Ref. [4], is axiomatic and motivated by the search of consistent equations to describe the back reaction of the quantum stress-energy fluctuations on the gravitational field. The second, based on Ref. [5], is by functional methods. It is motivated by the observation [1] that in some open quantum systems classicalization and decoherence [25] on the system may be brought about by interaction with an environment. The environment being in this case the matter fields and some degrees of freedom of the quantum gravitational field [27].

### A. Semiclassical gravity

Semiclassical gravity describes the interaction of the gravitational field assumed to be a classical field with quantum matter fields. This theory cannot be derived as a limit in a quantum theory of gravity, but can be formally derived in

several ways. One of them is the leading  $1/N$  approximation of quantum gravity interacting with  $N$  independent and identical free quantum fields [56]. When one assumes that the fields interact with gravity only and keep the value of  $NG$  finite, where  $G$  is Newton's gravitational constant, one arrives at a theory in which formally the gravitational field can be treated as a c-number (i.e. quantized at tree level) but matter fields are fully quantized.

In the semiclassical theory the expectation value of the stress-energy tensor in a given quantum state is the source of the gravitational field. This theory may be summarized as follows. Let  $(\mathcal{M}, g_{\mu\nu})$  be a globally hyperbolic four-dimensional spacetime with metric  $g_{\mu\nu}$  and consider a real scalar quantum field  $\phi$  of mass  $m$  propagating on it (we assume a scalar field for simplicity but this is not essential).

The classical action  $S_m$  for this matter field is given by the functional

$$S_m[g, \phi] = -\frac{1}{2} \int d^4x \sqrt{-g} [g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + (m^2 + \xi R) \phi^2], \quad (2.1)$$

where  $\nabla_\mu$  is the covariant derivative associated of the metric  $g_{\mu\nu}$ ,  $\xi$  is a coupling parameter coupling the field to the scalar curvature  $R$ , and  $g$  under the square root stands for the determinant of  $g_{\mu\nu}$ .

The field may be quantized in the manifold using the standard canonical formalism [20–22]. The field operator in the Heisenberg representation  $\hat{\phi}$  is an operator valued distribution solution of the Klein-Gordon equation, the field equation of Eq. (2.1),

$$(\square - m^2 - \xi R)\hat{\phi} = 0. \quad (2.2)$$

We will write the field operator as  $\hat{\phi}[g](x)$  to indicate that it is a functional of the metric.

The classical stress-energy tensor is obtained by functional derivation of this action in the usual way  $T^{\mu\nu}(x) = (2/\sqrt{-g})\delta S_m/\delta g_{\mu\nu}$ , leading to

$$\begin{aligned} T_{\mu\nu}[g, \phi] &= \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla^\rho \phi \nabla_\rho \phi + m^2 \phi^2) \\ &\quad + \xi (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu + G_{\mu\nu}) \phi^2, \end{aligned} \quad (2.3)$$

where  $\square = \nabla_\mu \nabla^\mu$  and  $G_{\mu\nu}$  is the Einstein tensor. With the notation  $T_{\mu\nu}[g, \phi]$  we explicitly indicate that the stress-energy tensor is a functional of the metric  $g_{\mu\nu}$  and the field  $\phi$ .

The next step is to define a stress-energy tensor operator  $\hat{T}_{\mu\nu}[g](x)$ . Naively one would replace the classical field  $\phi$  in the above functional by the quantum operator  $\hat{\phi}[g]$ , but this procedure involves taking the product of two distributions at the same spacetime point. This is ill-defined and we need a regularization procedure. There are several regularization methods which one may use, one is the point-splitting or point-separation regularization method in which one introduces a point  $y$  in a neighborhood of the point  $x$  and then uses as a regulator the vector tangent at the point  $x$  of the geodesic joining  $x$  and  $y$ ; this method is discussed in detail in section V. Another well known method is dimensional regularization in which one works in arbitrary  $n$  dimensions, where  $n$  is not necessarily an integer, and then uses as the regulator the parameter  $\epsilon = n - 4$ ; this method is used in this section and in section III. The regularized stress-energy operator using the Weyl ordering prescription, *i.e.* symmetrical ordering, can be written as

$$\hat{T}^{\mu\nu}[g] = \frac{1}{2} \{ \nabla^\mu \hat{\phi}[g], \nabla^\nu \hat{\phi}[g] \} + \mathcal{D}^{\mu\nu}[g] \hat{\phi}^2[g], \quad (2.4)$$

where  $\mathcal{D}^{\mu\nu}[g]$  is the differential operator

$$\mathcal{D}_x^{\mu\nu} \equiv \left( \xi - \frac{1}{4} \right) g^{\mu\nu}(x) \square_x + \xi (R^{\mu\nu}(x) - \nabla_x^\mu \nabla_x^\nu). \quad (2.5)$$

Note that if dimensional regularization is used, the field operator  $\hat{\phi}$  propagates in a  $n$ -dimensional spacetime. Once the regularization prescription has been introduced a renormalized and regularized stress-energy operator may be defined as

$$\hat{T}_{\mu\nu}^R[g](x) = \hat{T}_{\mu\nu}[g](x) + F_{\mu\nu}^C[g](x) \hat{I}, \quad (2.6)$$

where  $\hat{I}$  is the identity operator and  $F_{\mu\nu}^C[g]$  are some symmetric tensor counterterms which depend on the regulator and are local functionals of the metric. Here we are assuming that all these terms depend on the regulator, see Ref. [5] for details. These states can be chosen in such a way that for any pair of physically acceptable states, *i.e.* Hadamard states in the sense of Ref. [22],  $|\psi\rangle$  and  $|\varphi\rangle$  the matrix element  $\langle \psi | T_{\mu\nu}^R | \varphi \rangle$  defined as the limit of the previous expression

when the regulator takes the physical value, is finite and satisfies Wald's axioms [21,57]. These counterterms can be extracted from the singular part of a Schwinger-DeWitt series [21,58,59]. The choice of these counterterms is not unique but this ambiguity can be absorbed into the renormalized coupling constants which appear in the equations of motion for the gravitational field.

The *semiclassical Einstein equations* for the metric  $g_{\mu\nu}$  can then be written as

$$\frac{1}{8\pi G} (G_{\mu\nu}[g] + \Lambda g_{\mu\nu}) - 2(\alpha A_{\mu\nu} + \beta B_{\mu\nu})[g] = \langle \hat{T}_{\mu\nu}^R \rangle[g], \quad (2.7)$$

where  $\langle \hat{T}_{\mu\nu}^R \rangle[g]$  is the expectation value of the operator (2.6) after the regulator takes the physical value in some physically acceptable state of the field on  $(\mathcal{M}, g_{\mu\nu})$ . Note that both the stress tensor and the quantum state are functionals of the metric, hence the notation. The parameters  $G$ ,  $\Lambda$ ,  $\alpha$  and  $\beta$  are the renormalized coupling constants, respectively, the gravitational constant, the cosmological constant and two dimensionless coupling constants which are zero in the classical Einstein equations. These constants must be understood as the result of “dressing” the bare constants which appear in the classical action before renormalization. The values of these constants must be determined by experiment. The left hand side of Eq. (2.7) may be derived from the gravitational action

$$S_g[g] = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) + \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2 \right], \quad (2.8)$$

where  $C_{\mu\nu\rho\sigma}$  is the Weyl tensor. The tensors  $A_{\mu\nu}$  and  $B_{\mu\nu}$  come from the functional derivatives with respect to the metric of the terms quadratic in the curvature in Eq. (2.8), they are explicitly given by

$$\begin{aligned} A_{\mu\nu} = & \frac{1}{2} g_{\mu\nu} C_{\tau\epsilon\rho\sigma} C^{\tau\epsilon\rho\sigma} - 2R_{\mu\rho\sigma\tau} R_{\nu}^{\rho\sigma\tau} + 4R_{\mu\rho} R_{\nu}^{\rho} - \frac{2}{3} R R_{\mu\nu} \\ & - 2\Box R_{\mu\nu} + \frac{2}{3} \nabla_{\mu} \nabla_{\nu} R + \frac{1}{3} g_{\mu\nu} \Box R, \end{aligned} \quad (2.9)$$

$$B_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R^2 - 2R R_{\mu\nu} + 2\nabla_{\mu} \nabla_{\nu} R - 2g_{\mu\nu} \Box R, \quad (2.10)$$

where  $R_{\mu\nu\rho\sigma}$  and  $R_{\mu\nu}$  are the Riemann and Ricci tensors, respectively. These two tensors are, like the Einstein and metric tensors, symmetric and divergenceless:  $\nabla^{\mu} A_{\mu\nu} = 0 = \nabla^{\mu} B_{\mu\nu}$ . Note that a classical stress-energy can also be added to the right hand side of Eq. (2.7), but for simplicity we omit such a term.

A solution of semiclassical gravity consists of a spacetime  $(\mathcal{M}, g_{\mu\nu})$ , a quantum field operator  $\hat{\phi}[g]$  which satisfies (2.1), and a physically acceptable state  $|\psi\rangle[g]$  for this field, such that Eq. (2.7) is satisfied when the expectation value of the renormalized stress-energy operator is evaluated in this state.

For a free quantum field this theory is robust in the sense that it is consistent and fairly well understood. As long as the gravitational field is assumed to be described by a classical metric, the above semiclassical Einstein equations seems to be the only plausible dynamical equation for this metric: the metric couples to matter fields via the stress-energy tensor and for a given quantum state the only physically observable c-number stress energy-tensor that one can construct is the above renormalized expectation value. However, lacking a full quantum gravity theory the scope and limits of the theory are not so well understood. It is assumed that the semiclassical theory should break down at Planck scales, which is when simple order of magnitude estimates suggest that the quantum effects of gravity should not be ignored because the energy of a quantum fluctuation in a Planck size region, as determined by the Heisenberg uncertainty principle, is comparable to the gravitational energy of that fluctuation.

The theory should also break down when the fluctuations of the stress-energy tensor are large. This has been emphasized by Ford and collaborators [60,61]. It is less clear, however, how to quantify what a large fluctuation is, and different criteria have been proposed [61,62,7,8]. In Sec. IV we will discuss at length the issue of the validity of the semiclassical theory. One may illustrate the problem by the following example inspired in Ref. [60].

Let us assume a quantum state formed by an isolated system which consists of a superposition with equal amplitude of one configuration with mass  $M_1$  and another with mass  $M_2$ . The semiclassical theory as described by Eq. (2.7) predicts that the gravitational field of the system is produced by the averaged mass  $(M_1 + M_2)/2$ . However, one would expect that if we send a succession of test particles to probe the gravitational field of the above system half of the time they would react to the field of a mass  $M_1$  and half of the time to the field of a mass  $M_2$ . If the two masses differ substantially the two predictions are clearly different, note that the fluctuation in mass of the quantum state is of order of  $(M_1 - M_2)^2$ . Although this example is suggestive a word of caution should be said in order not to take it too literally. In fact, if the previous masses are macroscopic the quantum system decoheres very quickly [63] and instead of being described by a pure quantum state it is described by a density matrix which diagonalizes in a certain pointer basis. For observables associated to such a pointer basis the matrix density description is equivalent to that provided by a statistical ensemble. In any case, though, the results will differ from the semiclassical prediction.

## B. Axiomatic route to stochastic semiclassical gravity

### 1. The noise kernel

The purpose of stochastic semiclassical gravity is to go beyond the semiclassical theory and account for the fluctuations of the stress-energy operator. But first, we have to give a physical observable that describes these fluctuations. To lowest order, these fluctuations are obviously described by the following bi-tensor, constructed with the two-point correlation of the stress-energy operator,

$$4N_{\mu\nu\rho\sigma}(x, y) = \frac{1}{2} \langle \{ \hat{t}_{\mu\nu}(x), \hat{t}_{\rho\sigma}(y) \} \rangle [g], \quad (2.11)$$

where the curly brackets mean anticommutator, and where

$$\hat{t}_{\mu\nu}(x) \equiv \hat{T}_{\mu\nu}(x) - \langle \hat{T}_{\mu\nu}(x) \rangle. \quad (2.12)$$

This bi-tensor will be called *noise kernel* from now on. Note that we have defined it in terms of the unrenormalized stress-tensor operator  $\hat{T}_{\mu\nu}[g](x)$  on a given background metric  $g_{\mu\nu}$ , thus a regulator is implicitly assumed on the r.h.s. of Eq. (2.11). However, for a linear quantum field the above kernel is free of ultraviolet divergencies because the ultraviolet behaviour of  $\langle \hat{T}_{\mu\nu}(x) \hat{T}_{\rho\sigma}(y) \rangle$  is the same as that of  $\langle \hat{T}_{\mu\nu}(x) \rangle \langle \hat{T}_{\rho\sigma}(y) \rangle$ , thus following Eq. (2.6) one can replace  $\hat{T}_{\mu\nu}$  by the renormalized operator  $\hat{T}_{\mu\nu}^R$  in Eq. (2.11); an alternative proof of this result is given in section V. The noise kernel should be thought of as a distribution function, the limit of coincidence points has meaning only in the sense of distributions. An analysis of the noise kernel based on the point-separation method is given in section V.

The bi-tensor  $N_{\mu\nu\rho\sigma}(x, y)$  is real and positive-semidefinite, as a consequence of  $\hat{T}_{\mu\nu}^R$  being self-adjoint. A simple proof can be given as follows. Let  $|\psi\rangle$  be a given quantum state and let  $\hat{Q}$  be a selfadjoint operator,  $\hat{Q}^\dagger = \hat{Q}$ , then one can write  $\langle \psi | \hat{Q} \hat{Q} | \psi \rangle = \langle \psi | \hat{Q}^\dagger \hat{Q} | \psi \rangle = |\hat{Q}|\psi\rangle|^2 \geq 0$ . Now let  $\hat{t}(x)$  be a selfadjoint operator representing the operator  $\hat{t}_{\mu\nu}$  of Eq. (2.11) but where the coordinate  $x$  now carries also the tensorial indices of Eq. (2.11), then if we define  $\hat{Q} = \int dx f(x) \hat{t}(x)$  for an arbitrary well behaved “function”  $f(x)$  (which carries also tensorial indices), the previous inequality can be written as  $\int dx dy f(x) \langle \psi | \hat{t}(x) \hat{t}(y) | \psi \rangle f(y) \geq 0$ , which is the condition for the noise kernel to be positive semi-definite. Note that when we will consider, later on, the inverse kernel  $N_{\mu\nu\rho\sigma}^{-1}(x, y)$  we will implicitly assume that we are working in the subspace obtained from the eigenvectors which have strictly positive eigenvalues when we diagonalize the noise kernel.

### 2. The Einstein-Langevin equation

Our purpose now is to perturbatively modify the semiclassical theory. Thus we will assume that the background spacetime metric  $g_{\mu\nu}$  is a solution of the semiclassical Einstein Eqs. (2.7). The stress-energy tensor will generally have quantum fluctuations on that background spacetime. One would expect that these fluctuations will have some effect on the gravitational field and that this may now be described by  $g_{\mu\nu} + h_{\mu\nu}$ , where we will assume that  $h_{\mu\nu}$  is a linear perturbation to the background solution. The renormalized stress-energy operator and the state of the quantum field may now be denoted by  $\hat{T}_{\mu\nu}^R[g + h]$  and  $|\psi\rangle[g + h]$ , respectively, and  $\langle \hat{T}_{\mu\nu}^R \rangle[g + h]$  will be the corresponding expectation value.

Let us now introduce a Gaussian stochastic tensor field  $\xi_{\mu\nu}$  defined by the following correlators:

$$\langle \xi_{\mu\nu}(x) \rangle_s = 0, \quad \langle \xi_{\mu\nu}(x) \xi_{\rho\sigma}(y) \rangle_s = N_{\mu\nu\rho\sigma}(x, y), \quad (2.13)$$

where  $\langle \dots \rangle_s$  means statistical average. The symmetry and positive semi-definite property of the noise kernel guarantees that the stochastic field tensor  $\xi_{\mu\nu}$  just introduced is well defined. Note that this stochastic tensor does not capture the whole quantum nature of the fluctuations of the stress-energy operator since it assumes that cumulants of higher order are zero, but it does so at the Gaussian level.

An important property of this stochastic tensor is that it is covariantly conserved in the background spacetime  $\nabla^\mu \xi_{\mu\nu} = 0$ . In fact, as a consequence of the conservation of  $\hat{T}_{\mu\nu}^R[g]$  one can see that  $\nabla_x^\mu N_{\mu\nu\rho\sigma}(x, y) = 0$ . Taking the divergence in Eq. (2.13) one can then show that  $\langle \nabla^\mu \xi_{\mu\nu} \rangle_s = 0$  and  $\langle \nabla_x^\mu \xi_{\mu\nu}(x) \nabla_y^\rho \xi_{\rho\sigma}(y) \rangle_s = 0$  so that  $\nabla^\mu \xi_{\mu\nu}$  is deterministic and represents with certainty the zero vector field in  $\mathcal{M}$ .

One can also see that for a conformal field, i.e. a field whose classical action is conformally invariant,  $\xi_{\mu\nu}$  is traceless, i.e.  $g^{\mu\nu} \xi_{\mu\nu} = 0$ , so that for a conformal matter field the stochastic source gives no correction to the trace anomaly.

In fact, from the trace anomaly result which states that  $g^{\mu\nu}\hat{T}_{\mu\nu}^R[g]$  is in this case a local c-number functional of  $g_{\mu\nu}$  times the identity operator, we have that  $g^{\mu\nu}(x)N_{\mu\nu\rho\sigma}(x,y)=0$ . It then follows from Eq. (2.13) that  $\langle g^{\mu\nu}\xi_{\mu\nu}\rangle_s=0$  and  $\langle g^{\mu\nu}(x)\xi_{\mu\nu}(x)g^{\rho\sigma}(y)\xi_{\rho\sigma}(y)\rangle_s=0$ ; an alternative proof based on the point-separation method is given in section V.

All these properties make it quite natural to incorporate into the Einstein equations the stress-energy fluctuations by using the stochastic tensor  $\xi_{\mu\nu}$  as the source of the metric perturbations. Thus we will write the following equation.

$$\frac{1}{8\pi G}(G_{\mu\nu}[g+h]+\Lambda(g_{\mu\nu}+h_{\mu\nu}))-2(\alpha A_{\mu\nu}+\beta B_{\mu\nu})[g+h]=\langle\hat{T}_{\mu\nu}^R\rangle[g+h]+2\xi_{\mu\nu}. \quad (2.14)$$

This equation known as the *semiclassical Einstein-Langevin equation*, is a dynamical equation for the metric perturbation  $h_{\mu\nu}$  to linear order. It describes the back-reaction of the metric to the quantum fluctuations of the stress-energy tensor of matter fields, and gives a first order correction to semiclassical gravity as described by the semiclassical Einstein equation (2.7). Note that the stochastic source  $\xi_{\mu\nu}$  is not dynamical, it is independent of  $h_{\mu\nu}$  since it describes the fluctuations of the stress tensor on the semiclassical background  $g_{\mu\nu}$ .

An important property of the Einstein-Langevin equation is that it is gauge invariant. In the sense that if we change  $h_{\mu\nu}\rightarrow h'_{\mu\nu}+\nabla_\mu\eta_\nu+\nabla_\nu\eta_\mu$  where  $\eta^\mu$  is a stochastic vector field on the manifold  $\mathcal{M}$ . All the tensor which appear in the Eq. (2.14) transform as  $R_{\mu\nu}[g+h']=R_{\mu\nu}[g+h]+\mathcal{L}_\eta R_{\mu\nu}[g]$  to linear order in the perturbations, where  $\mathcal{L}_\eta$  is the Lie derivative with respect to  $\eta^\mu$ . If we substitute  $h$  by  $h'$  in Eq. (2.14), we get Eq. (2.14) plus the Lie derivative of the combination of the tensor which appear in Eq. (2.7). This last combination vanishes when Eq. (2.7) is satisfied, as we have assumed. It is thus necessary that the background metric  $g_{\mu\nu}$  be a solution of semiclassical gravity.

A solution of Eq. (2.14) can be formally written as  $h_{\mu\nu}[\xi]$ . This solution is characterized by the whole family of its correlation functions. From the statistical average of this equation we have that  $g_{\mu\nu}+\langle h_{\mu\nu}\rangle_s$  must be a solution of the semiclassical Einstein equation linearized around the background  $g_{\mu\nu}$ . The fluctuation of the metric around this average are described by the moments of the stochastic field  $h_{\mu\nu}^s[\xi]=h_{\mu\nu}[\xi]-\langle h_{\mu\nu}\rangle_s$ . Thus the solutions of the Einstein-Langevin equation will provide the two point metric correlators  $\langle h_{\mu\nu}^s(x)h_{\rho\sigma}^s(y)\rangle_s$ .

The stochastic theory may be understood as an intermediate step between the semiclassical theory and the full quantum theory. In the sense that whereas the semiclassical theory depends on the point-like value of the stress-energy operator, the stochastic theory carries information also on the two-point correlation of the stress-energy operator.

We should also emphasize that, even if the metric fluctuations are classical (stochastic), their origin is quantum not only because they are induced by the fluctuations of quantum matter, but also because they are supposed to describe some remnants of the quantum gravity fluctuations after some mechanism for decoherence and classicalization of the metric field [64–68]. From the formal assumption that such a mechanism is the Gell-Mann and Hartle mechanism of environment-induced decoherence of suitably coarse-grained system variables [64,65], one may, in fact, derive the stochastic semiclassical theory [69]. Nevertheless, that derivation is of course formal, given that, due to the lack of the full quantum theory of gravity, the classicalization mechanism for the gravitational field is not understood.

### 3. A toy model

To illustrate the relation between the quantum and stochastic descriptions it is useful to introduce the following toy model. Let us assume that the gravitational equations are described by a linear field  $h$  coupled to a scalar source  $T[\phi^2]$  independent of  $h$ , with the classical equations  $\square h=T$  in flat spacetime. We should emphasize that this model would describe a linear theory of gravity analogous to electromagnetism and does not mimic the linearized theory of gravity in which  $T$  is also linear in  $h$ . However it captures some of the key features of linearized gravity.

In the Heisenberg representation the quantum field  $\hat{h}$  satisfies

$$\square\hat{h}=\hat{T}. \quad (2.15)$$

The solutions of this equation, i.e. the field operator at the point  $x$ ,  $\hat{h}_x$ , may be written in terms of the retarded propagator  $G_{xy}$  as,

$$\hat{h}_x=\hat{h}_x^0+\int dx'G_{xx'}\hat{T}_{x'}, \quad (2.16)$$

where  $h^0$  is the free field. From this solution we may compute, for instance, the following two point quantum correlation function (the anticommutator)

$$\frac{1}{2}\langle\{\hat{h}_x,\hat{h}_y\}\rangle=\frac{1}{2}\langle\{\hat{h}_x^0,\hat{h}_y^0\}\rangle+\frac{1}{2}\int\int dx'dy'G_{xx'}G_{yy'}\langle\{\hat{T}_{x'},\hat{T}_{y'}\}\rangle, \quad (2.17)$$

where the expectation value is taken in the quantum state in which both fields  $\phi$  and  $h$  have been quantized and we have used that for the free field  $\langle \hat{h}^0 \rangle = 0$ .

Let us now compare with the stochastic theory for this problem. As discussed previously we should write a classical equation for the field  $h$  where on the r.h.s. we substitute  $\hat{T}$  by the expectation value  $\langle \hat{T} \rangle$  plus a Gaussian stochastic field  $\xi$  defined by the quantum correlation function of  $T$  (the noise kernel). Note that in our model since  $T$  is independent of  $h$  we may simply renormalize the expectation value using time ordering, then for the vacuum state of the field  $\hat{\phi}$ , we would simply have  $\langle \hat{T} \rangle_0 = 0$ . Thus  $\xi$  is defined as in (2.13) by  $\langle \xi \rangle_s = 0$  and  $\langle \xi_x \xi_y \rangle_s = (1/2) \langle \{\hat{T}_x, \hat{T}_y\} \rangle - \langle \hat{T}_x \rangle \langle \hat{T}_y \rangle$ . The semiclassical stochastic equation is thus

$$\square h = \langle T \rangle + \xi. \quad (2.18)$$

The solution of this equation may be written in terms of the retarded propagator as,

$$h_x = h_x^0 + \int dx' G_{xx'} \left( \langle \hat{T}_{x'} \rangle + \xi_{x'} \right), \quad (2.19)$$

from where the two point correlation function for the classical field  $h$ , after using the definition of  $\xi$  and that  $\langle h^0 \rangle_s = 0$ , is given by

$$\langle h_x h_y \rangle_s = \langle h_x^0 h_y^0 \rangle_s + \frac{1}{2} \int dx' dy' G_{xx'} G_{yy'} \langle \{\hat{T}_{x'}, \hat{T}_{y'}\} \rangle. \quad (2.20)$$

Comparing (2.17) with (2.20) we see that the respective second terms on the right hand side are identical provided the expectation values are computed in the same quantum state for the field  $\hat{\phi}$ , note that we have assumed that  $T$  does not depend on  $h$ . The fact that the field  $h$  is also quantized in (2.17) does not change the previous statement. In the real theory of gravity  $T$ , in fact, depends also on  $h$  and then the previous statement is only true approximately, i.e perturbatively in  $h$ . The nature of the first terms on the right hand sides of equations (2.17) and (2.20) is different: in the first case it is the two point expectation value of the free quantum field  $\hat{h}_0$  whereas in the second case it is the average of the two point classical average of the homogeneous field  $h_0$ , which depends on the initial conditions. Now we can still make these terms to be equal to each other if we assume for the homogeneous field  $h$  a distribution of initial conditions such that  $\langle h_x^0 h_y^0 \rangle_s = (1/2) \langle \{\hat{h}_x^0, \hat{h}_y^0\} \rangle$ . Thus, under this assumption on initial conditions for the field  $h$  the two point correlation function of (2.20) equal the quantum expectation value of (2.17) exactly.

Thus in a linear theory as in the model just described one may just use the statistical description given by (2.18) to compute the quantum two point function of equation (2.16). This does not mean that we can recover all quantum correlation functions with the stochastic description, see Ref. [70] for a general discussion about this point. Note that, for instance, the commutator of the classical stochastic field  $h$  is obviously zero, but the commutator of the quantum field  $\hat{h}$  is not zero for timelike separated points; this is the prize we pay for the introduction of the classical field  $\xi$  to describe the quantum fluctuations. Furthermore, the statistical description is not able to account for the graviton-graviton effects which go beyond the linear approximation in  $\hat{h}$ .

### C. Functional approach

The Einstein-Langevin equation (2.14) may also be derived by a method based on functional techniques [5]. Functional techniques have a long history in semiclassical gravity. It started when effective action methods, which are so familiar in quantum field theory, were used to study the back-reaction of quantum fields in cosmological models [71]. These methods were of great help in the study of cosmological anisotropies since they allowed the introduction of familiar perturbative treatments into the subject. The most common method, the in-out effective action method, however, led to equations of motion which were not real because they were tailored to compute transition elements of quantum operators rather than expectation values. Fortunately the appropriate technique had already been developed by Schwinger and Keldysh [12] in the so called Closed Time Path (CTP) or in-in effective action. These techniques were soon adapted to the gravitational context [13] and were applied to different problems in cosmology. As a result one was now able to deduce the semiclassical Einstein equations by the CTP functional method: starting with an action for the interaction of gravity with matter fields, treating the matter fields as quantum fields and the gravitational field at tree level only.

Furthermore, in this case the CTP functional formalism turns out to be related [31,72–75,5] to the influence functional formalism of Feynman and Vernon [76] since the full quantum system may be understood as consisting of a distinguished subsystem (the “system” of interest) interacting with an environment (the remaining degrees of



freedom). The integration of the environment variables in a CTP path integral yields the influence functional, from which one can define an effective action for the dynamics of the system [77,31,30,78,79,72].

In our case, we consider the metric field  $g_{\mu\nu}(x)$  as the “system” degrees of freedom, and the scalar field  $\phi(x)$  and also some “high-momentum” gravitational modes [68] as the “environment” variables. Unfortunately, since the form of a complete quantum theory of gravity interacting with matter is unknown, we do not know what these “high-momentum” gravitational modes are. Such a fundamental quantum theory might not even be a field theory, in which case the metric and scalar fields would not be fundamental objects [3]. Thus, in this case, we cannot attempt to evaluate the influence action of Feynman and Vernon starting from the fundamental quantum theory and performing the path integrations in the environment variables. Instead, we introduce the influence action for an effective quantum field theory of gravity and matter [80,30], in which such “high-momentum” gravitational modes are assumed to have been already “integrated out.” Adopting the usual procedure of effective field theories [81,80], one has to take the effective action for the metric and the scalar field of the most general local form compatible with general covariance:

$$S[g, \phi] \equiv S_g[g] + S_m[g, \phi] + \dots,$$

where  $S_g[g]$  and  $S_m[g, \phi]$  are given by Eqs. (2.8) and (2.1), respectively, and the dots stand for terms of order higher than two in the curvature and in the number of derivatives of the scalar field. Here, we shall neglect the higher order terms as well as self-interaction terms for the scalar field. The second order terms are necessary to renormalize one-loop ultraviolet divergencies of the scalar field stress-energy tensor, as we have already seen. Since  $\mathcal{M}$  is a globally hyperbolic manifold, we can foliate it by a family of  $t = \text{constant}$  Cauchy hypersurfaces  $\Sigma_t$ . We denote by  $\mathbf{x}$  the coordinates on each of these hypersurfaces, and by  $t_i$  and  $t_f$  some initial and final times, respectively. The integration domain for the action terms must be understood as a compact region  $\mathcal{U}$  of the manifold  $\mathcal{M}$ , bounded by the hypersurfaces  $\Sigma_{t_i}$  and  $\Sigma_{t_f}$ .

Assuming the form (2.1) for the effective action which couples the scalar and the metric fields, we can now introduce the corresponding influence functional. This is a functional of two copies of the metric field that we denote by  $g_{\mu\nu}^+$  and  $g_{\mu\nu}^-$ . Let us assume that, in the quantum effective theory, the state of the full system (the scalar and the metric fields) in the Schrödinger picture at the initial time  $t = t_i$  can be described by a factorizable density operator, *i.e.*, a density operator which can be written as the tensor product of two operators on the Hilbert spaces of the metric and of the scalar field. Let  $\hat{\rho}^s(t_i)$  be the density operator describing the initial state of the scalar field. If we consider the theory of a scalar field quantized in the classical background spacetime  $(\mathcal{M}, g_{\mu\nu})$  through the action (2.1), a state in the Heisenberg representation described by a density operator  $\hat{\rho}[g]$  corresponds to this state. Let  $\{|\varphi(\mathbf{x})\rangle^s\}$  be the basis of eigenstates of the scalar field operator  $\hat{\phi}^s(\mathbf{x})$  in the Schrödinger representation:  $\hat{\phi}^s(\mathbf{x})|\varphi\rangle^s = \varphi(\mathbf{x})|\varphi\rangle^s$ . The matrix elements of  $\hat{\rho}^s(t_i)$  in this basis will be written as  $\rho_i[\varphi, \tilde{\varphi}] \equiv {}^s\langle\varphi|\hat{\rho}^s(t_i)|\tilde{\varphi}\rangle^s$ . We can now introduce the *influence functional* as the following path integral over two copies of the scalar field:

$$\mathcal{F}_{\text{IF}}[g^+, g^-] \equiv \int \mathcal{D}[\phi_+] \mathcal{D}[\phi_-] \rho_i[\phi_+(t_i), \phi_-(t_i)] \delta[\phi_+(t_f) - \phi_-(t_f)] e^{i(S_m[g^+, \phi_+] - S_m[g^-, \phi_-])}. \quad (2.21)$$

The above double path integral can be rewritten as a closed time path (CTP) integral, namely, as an integral over a single copy of field paths with two different time branches, one going forward in time from  $t_i$  to  $t_f$ , and the other going backward in time from  $t_f$  to  $t_i$ . From this influence functional, the *influence action* of Feynman and Vernon,  $S_{\text{IF}}[g^+, g^-]$ , is defined by

$$\mathcal{F}_{\text{IF}}[g^+, g^-] \equiv e^{iS_{\text{IF}}[g^+, g^-]}, \quad (2.22)$$

this action has all the relevant information on the matter fields. Then we can define the *effective action* for the gravitational field,  $S_{\text{eff}}[g^+, g^-]$ , as

$$S_{\text{eff}}[g^+, g^-] \equiv S_g[g^+] - S_g[g^-] + S_{\text{IF}}[g^+, g^-]. \quad (2.23)$$

This is the action for the classical gravitational field in the CTP formalism. However, since the gravitational field is only treated at tree level, this is also the effective “classical” action from where the classical equations can be derived.

Expression (2.21) is ill-defined, it must be regularized to get a meaningful influence functional. We shall assume that we can use dimensional regularization, that is, that we can give sense to Eq. (2.21) by dimensional continuation of all the quantities that appear in this expression. For this we substitute the action  $S_m$  in (2.21) by some generalization to  $n$  spacetime dimensions, and similarly for  $S_g[g]$ . In this last case the parameters  $G$ ,  $\Lambda$ ,  $\alpha$  and  $\beta$  of Eq. (2.8) are the bare parameters  $G_B$ ,  $\Lambda_B$ ,  $\alpha_B$  and  $\beta_B$ , and instead of the square of the Weyl tensor one must use  $(2/3)(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu})$  which by the Gauss-Bonnet theorem leads the same equations of motion as the action (2.8) when  $n = 4$ . The form of  $S_g$  in  $n$  dimensions is suggested by the Schwinger-DeWitt analysis of the ultraviolet divergencies in the

matter stress-energy tensor using dimensional regularization. One can then write the effective action of Feynman and Vernon,  $S_{\text{eff}}[g^+, g^-]$ , in Eq. (2.23) in dimensional regularization. Since both  $S_m$  and  $S_g$  contain second order derivatives of the metric, one should also add some boundary terms [19,30]. The effect of these terms is to cancel out the boundary terms which appear when taking variations of  $S_{\text{eff}}[g^+, g^-]$  keeping the value of  $g_{\mu\nu}^+$  and  $g_{\mu\nu}^-$  fixed on the boundary of  $\mathcal{U}$ . Alternatively, in order to obtain the equations of motion for the metric in the semiclassical regime, we can work with the action terms without boundary terms and neglect all boundary terms when taking variations with respect to  $g_{\mu\nu}^\pm$ . From now on, all the functional derivatives with respect to the metric will be understood in this sense.

Now we can derive the semiclassical Einstein equations (2.7). Using the definition of the stress-energy tensor  $T^{\mu\nu}(x) = (2/\sqrt{-g})\delta S_m/\delta g_{\mu\nu}$  and the definition of the influence functional, Eqs. (2.21) and (2.22), we see that

$$\langle \hat{T}^{\mu\nu}(x) \rangle[g] = \frac{2}{\sqrt{-g(x)}} \frac{\delta S_{\text{IF}}[g^+, g^-]}{\delta g_{\mu\nu}^+(x)} \Big|_{g^+=g^-=g}, \quad (2.24)$$

where the expectation value is taken in the  $n$ -dimensional spacetime generalization of the state described by  $\hat{\rho}[g]$ . Therefore, differentiating  $S_{\text{eff}}[g^+, g^-]$  in Eq. (2.23) with respect to  $g_{\mu\nu}^+$ , and then setting  $g_{\mu\nu}^+ = g_{\mu\nu}^- = g_{\mu\nu}$ , we get the semiclassical Einstein equation in  $n$  dimensions. This equation is then renormalized by absorbing the divergencies in the regularized  $\langle \hat{T}^{\mu\nu} \rangle[g]$  into the bare parameters, and taking the limit  $n \rightarrow \infty$  we get the physical semiclassical Einstein equations (2.7).

#### D. Influence functional route to stochastic semiclassical gravity

In this subsection we derive the semiclassical Einstein-Langevin equation (2.14) by means of the influence functional. We also work out the semiclassical Einstein-Langevin equations more explicitly, in a form more suitable for specific calculations.

In the spirit of the previous derivation of the Einstein-Langevin equations, we now seek a dynamical equation for a linear perturbation  $h_{\mu\nu}$  to a semiclassical metric  $g_{\mu\nu}$ , solution of Eq. (2.7). Strictly speaking if we use dimensional regularization we must consider the  $n$ -dimensional version of that equation; see Ref. [5] for details. From the result of the previous subsection, if such equation were simply a linearized semiclassical Einstein equation, it could be obtained from an expansion of the effective action  $S_{\text{eff}}[g + h^+, g + h^-]$ . In particular, since, from Eq. (2.24), we have that

$$\langle \hat{T}_n^{\mu\nu}(x) \rangle[g + h] = \frac{2}{\sqrt{-\det(g+h)(x)}} \frac{\delta S_{\text{IF}}[g + h^+, g + h^-]}{\delta h_{\mu\nu}^+(x)} \Big|_{h^+=h^-=h}, \quad (2.25)$$

the expansion of  $\langle \hat{T}_n^{\mu\nu} \rangle[g + h]$  to linear order in  $h_{\mu\nu}$  can be obtained from an expansion of the influence action  $S_{\text{IF}}[g + h^+, g + h^-]$  up to second order in  $h_{\mu\nu}^\pm$ .

To perform the expansion of the influence action, we have to compute the first and second order functional derivatives of  $S_{\text{IF}}[g^+, g^-]$  and then set  $g_{\mu\nu}^+ = g_{\mu\nu}^- = g_{\mu\nu}$ . If we do so using the path integral representation (2.21), we can interpret these derivatives as expectation values of operators. The relevant second order derivatives are

$$\begin{aligned} \frac{1}{\sqrt{-g(x)}\sqrt{-g(y)}} \frac{\delta^2 S_{\text{IF}}[g^+, g^-]}{\delta g_{\mu\nu}^+(x) \delta g_{\rho\sigma}^+(y)} \Big|_{g^+=g^-=g} &= -H_S^{\mu\nu\rho\sigma}[g](x, y) - K^{\mu\nu\rho\sigma}[g](x, y) + iN^{\mu\nu\rho\sigma}[g](x, y), \\ \frac{1}{\sqrt{-g(x)}\sqrt{-g(y)}} \frac{\delta^2 S_{\text{IF}}[g^+, g^-]}{\delta g_{\mu\nu}^+(x) \delta g_{\rho\sigma}^-(y)} \Big|_{g^+=g^-=g} &= -H_A^{\mu\nu\rho\sigma}[g](x, y) - iN^{\mu\nu\rho\sigma}[g](x, y), \end{aligned} \quad (2.26)$$

where

$$N^{\mu\nu\rho\sigma}[g](x, y) \equiv \frac{1}{8} \langle \{ \hat{t}^{\mu\nu}(x), \hat{t}^{\rho\sigma}(y) \} \rangle [g], \quad H_S^{\mu\nu\rho\sigma}[g](x, y) \equiv \frac{1}{4} \text{Im} \left\langle T^* \left( \hat{T}^{\mu\nu}(x) \hat{T}^{\rho\sigma}(y) \right) \right\rangle [g],$$

$$H_A^{\mu\nu\rho\sigma}[g](x, y) \equiv -\frac{i}{8} \left\langle \left[ \hat{T}^{\mu\nu}(x), \hat{T}^{\rho\sigma}(y) \right] \right\rangle [g], \quad K^{\mu\nu\rho\sigma}[g](x, y) \equiv \frac{-1}{\sqrt{-g(x)}\sqrt{-g(y)}} \left\langle \frac{\delta^2 S_m[g, \phi]}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)} \Big|_{\phi=\hat{\phi}} \right\rangle [g],$$

with  $\hat{t}^{\mu\nu}$  defined in Eq. (2.12), and where we use a Weyl ordering prescription for the operators in the last of these expressions. Here,  $[ , ]$  means the commutator, and we use the symbol  $T^*$  to denote that, first, we have to time order

the field operators  $\hat{\phi}$  and then apply the derivative operators which appear in each term of the product  $T^{\mu\nu}(x)T^{\rho\sigma}(y)$ , where  $T^{\mu\nu}$  is the functional (2.3). This  $T^*$  “time ordering” arises because we have path integrals containing products of derivatives of the field, which can be expressed as derivatives of the path integrals which do not contain such derivatives. Notice, from their definitions, that all the kernels which appear in expressions (2.26) are real and also that  $H_A^{\mu\nu\rho\sigma}$  is free of ultraviolet divergencies in the limit  $n \rightarrow 4$ .

From Eqs. (2.26) it is clear that the imaginary part of the influence action, which does not contribute to the semiclassical Einstein equation (2.7) because the expectation value of  $\hat{T}^{\mu\nu}[g]$  is real, contains information on the fluctuations of this operator. From (2.24) and (2.26), taking into account that  $S_{\text{IF}}[g, g] = 0$  and that  $S_{\text{IF}}[g^-, g^+] = -S_{\text{IF}}^*[g^+, g^-]$ , we can write the expansion for the influence action  $S_{\text{IF}}[g+h^+, g+h^-]$  around a background metric  $g_{ab}$  in terms of the previous kernels. Taking into account that these kernels satisfy the symmetry relations

$$H_S^{\mu\nu\rho\sigma}(x, y) = H_S^{\rho\sigma\mu\nu}(y, x), \quad H_A^{\mu\nu\rho\sigma}(x, y) = -H_A^{\rho\sigma\mu\nu}(y, x), \quad K^{\mu\nu\rho\sigma}(x, y) = K^{\rho\sigma\mu\nu}(y, x), \quad (2.27)$$

and introducing the new kernel

$$H^{\mu\nu\rho\sigma}(x, y) \equiv H_S^{\mu\nu\rho\sigma}(x, y) + H_A^{\mu\nu\rho\sigma}(x, y), \quad (2.28)$$

the expansion of  $S_{\text{IF}}$  can be finally written as

$$\begin{aligned} S_{\text{IF}}[g+h^+, g+h^-] &= \frac{1}{2} \int d^4x \sqrt{-g(x)} \langle \hat{T}^{\mu\nu}(x) \rangle [g] [h_{\mu\nu}(x)] \\ &\quad - \frac{1}{2} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} [h_{\mu\nu}(x)] (H^{\mu\nu\rho\sigma}[g](x, y) + K^{\mu\nu\rho\sigma}[g](x, y)) \{h_{\rho\sigma}(y)\} \\ &\quad + \frac{i}{2} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} [h_{\mu\nu}(x)] N^{\mu\nu\rho\sigma}[g](x, y) [h_{\rho\sigma}(y)] + 0(h^3), \end{aligned} \quad (2.29)$$

where we have used the notation

$$[h_{\mu\nu}] \equiv h_{\mu\nu}^+ - h_{\mu\nu}^-, \quad \{h_{\mu\nu}\} \equiv h_{\mu\nu}^+ + h_{\mu\nu}^-. \quad (2.30)$$

We are now in the position to carry out the formal derivation of the semiclassical Einstein-Langevin equation. The procedure is well known [31,30,73,82], it consists of deriving a new “stochastic” effective action using the the following identity:

$$e^{-\frac{1}{2} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} [h_{\mu\nu}(x)] N^{\mu\nu\rho\sigma}(x, y) [h_{\rho\sigma}(y)]} = \int \mathcal{D}[\xi] \mathcal{P}[\xi] e^{i \int d^4x \sqrt{-g(x)} \xi^{\mu\nu}(x) [h_{\mu\nu}(x)]}, \quad (2.31)$$

where  $\mathcal{P}[\xi]$  is the probability distribution functional of a Gaussian stochastic tensor  $\xi^{\mu\nu}$  characterized by the correlators (2.13) with  $N^{abcd}$  given by Eq. (2.11), and where the path integration measure is assumed to be a scalar under diffeomorphisms of  $(\mathcal{M}, g_{\mu\nu})$ . The above identity follows from the identification of the right hand side of (2.31) with the characteristic functional for the stochastic field  $\xi^{\mu\nu}$ . In fact, by differentiation of this expression with respect to  $[h_{\mu\nu}]$ , it can be checked that this is the characteristic functional of a stochastic field characterized by the correlators (2.13). When  $N^{\mu\nu\rho\sigma}(x, y)$  is strictly positive definite, the probability distribution functional for  $\xi^{\mu\nu}$  is explicitly given by

$$\mathcal{P}[\xi] = \frac{e^{-\frac{1}{2} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} \xi^{\mu\nu}(x) N_{\mu\nu\rho\sigma}^{-1}(x, y) \xi^{\rho\sigma}(y)}}{\int \mathcal{D}[\xi] e^{-\frac{1}{2} \int d^4z d^4w \sqrt{-g(z)} \sqrt{-g(w)} \xi^{\tau\omega}(z) N_{\tau\omega\epsilon\kappa}^{-1}(z, w) \xi^{\epsilon\kappa}(w)}}, \quad (2.32)$$

where  $N_{\mu\nu\rho\sigma}^{-1}[g](x, y)$  is the inverse of  $N^{\mu\nu\rho\sigma}[g](x, y)$  defined by

$$\int d^4z \sqrt{-g(z)} N^{\mu\nu\tau\omega}(x, z) N_{\tau\omega\rho\sigma}^{-1}(z, y) = \frac{1}{2} (\delta_\rho^\mu \delta_\sigma^\nu + \delta_\sigma^\mu \delta_\rho^\nu) \frac{\delta^4(x-y)}{\sqrt{-g(x)}}. \quad (2.33)$$

We may now introduce the *stochastic effective action* as

$$S_{\text{eff}}^s[h^+, h^-; g; \xi] \equiv S_g[g+h^+] - S_g[g+h^-] + S_{\text{IF}}^s[h^+, h^-; g; \xi], \quad (2.34)$$

where the “stochastic” influence action is defined as

$$S_{\text{IF}}^s[h^+, h^-; g; \xi_n] \equiv \text{Re } S_{\text{IF}}[g+h^+, g+h^-] + \int d^4x \sqrt{-g(x)} \xi^{\mu\nu}(x) [h_{\mu\nu}(x)] + 0(h^3). \quad (2.35)$$

Note that the influence functional as defined from the influence action (2.29) can be written as a statistical average over  $\xi^{\mu\nu}$ :

$$\mathcal{F}_{\text{IF}}[g+h^+, g+h^-] = \left\langle e^{iS_{\text{IF}}^s[h^+, h^-; g; \xi_n]} \right\rangle_s. \quad (2.36)$$

Thus, the effect of the imaginary part of the influence action (2.29) on the corresponding influence functional is equivalent to the averaged effect of the stochastic source  $\xi^{\mu\nu}$  coupled linearly to the perturbations  $h_{\mu\nu}^\pm$ .

The stochastic equations of motion for  $h_{\mu\nu}$  can now be derived as

$$\frac{1}{\sqrt{-\det(g+h)(x)}} \frac{\delta S_{\text{eff}}^s[h^+, h^-; g; \xi_n]}{\delta h_{ab}^+(x)} \Big|_{h^+=h^-=h} = 0. \quad (2.37)$$

Then, from (2.25), taking into account that only the real part of the influence action contributes to the expectation value of the stress-energy tensor, we get, to linear order in  $h_{\mu\nu}$  the stochastic semiclassical equations (2.14). To be precise we get first the regularized  $n$ -dimensional equations with the bare parameters, and where instead of the tensor  $A^{\mu\nu}$  we get  $(2/3)D^{\mu\nu}$  where

$$\begin{aligned} D^{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{-g} (R_{\rho\sigma\tau\omega} R^{\rho\sigma\tau\omega} - R_{\rho\sigma} R^{\rho\sigma}) \\ &= \frac{1}{2} g^{\mu\nu} (R_{\rho\sigma\tau\omega} R^{\rho\sigma\tau\omega} - R_{\rho\sigma} R^{\rho\sigma} + \square R) - 2R^{\mu\rho\sigma\tau} R^\nu_{\rho\sigma\tau} \\ &\quad - 2R^{\mu\rho\nu\sigma} R_{\rho\sigma} + 4R^{\mu\rho} R_\rho^\nu - 3\square R^{\mu\nu} + \nabla^\mu \nabla^\nu R. \end{aligned} \quad (2.38)$$

Of course, when  $n=4$  these tensors are related,  $A^{\mu\nu} = (2/3)D^{\mu\nu}$ . After that we renormalize and take the limit  $n \rightarrow 4$  to obtain the Einstein-Langevin equations in the physical spacetime.

### 1. Explicit linear form of the Einstein-Langevin equation

We can write the Einstein-Langevin equation in a more explicit form by working out the expansion of  $\langle \hat{T}^{\mu\nu} \rangle[g+h]$  up to linear order in the perturbation  $h_{\mu\nu}$ . From Eq. (2.25), we see that this expansion can be easily obtained from (2.29). The result is

$$\langle \hat{T}_n^{\mu\nu}(x) \rangle[g+h] = \langle \hat{T}_n^{\mu\nu}(x) \rangle[g] + \langle \hat{T}_n^{(1)\mu\nu}[g; h](x) \rangle[g] - 2 \int d^n y \sqrt{-g(y)} H_n^{\mu\nu\rho\sigma}[g](x, y) h_{\rho\sigma}(y) + 0(h^2), \quad (2.39)$$

where the operator  $\hat{T}_n^{(1)\mu\nu}$  is defined from the term of first order in the expansion of  $T^{ab}[g+h]$  as

$$T^{\mu\nu}[g+h] = T^{\mu\nu}[g] + T^{(1)\mu\nu}[g; h] + 0(h^2), \quad (2.40)$$

using, as always, a Weyl ordering prescription for the operators in the last definition. Here we use a subscript  $n$  on a given tensor to indicate that we are explicitly working in  $n$ -dimensions, as we use dimensional regularization, and we also use the superindex (1) to generally indicate that the tensor is the first order correction, linear in  $h_{\mu\nu}$ , in a perturbative expansion around the background  $g_{\mu\nu}$ .

Using the Klein-Gordon equation (2.2), and expressions (2.3) for the stress-energy tensor and the corresponding operator operator, we can write

$$\hat{T}_n^{(1)\mu\nu}[g; h] = \left( \frac{1}{2} g^{\mu\nu} h_{\rho\sigma} - \delta_\rho^\mu h_\sigma^\nu - \delta_\rho^\nu h_\sigma^\mu \right) \hat{T}_n^{\rho\sigma}[g] + \mathcal{F}^{\mu\nu}[g; h] \hat{\phi}_n^2[g], \quad (2.41)$$

where  $\mathcal{F}^{\mu\nu}[g; h]$  is the differential operator

$$\begin{aligned} \mathcal{F}^{\mu\nu} &\equiv \left( \xi - \frac{1}{4} \right) \left( h^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h^\rho_\rho \right) \square + \frac{\xi}{2} \left[ \nabla^\rho \nabla^\mu h_\rho^\nu + \nabla^\rho \nabla^\nu h_\rho^\mu - \square h^{\mu\nu} - \nabla^\mu \nabla^\nu h^\rho_\rho - g^{\mu\nu} \nabla^\rho \nabla^\sigma h_{\rho\sigma} \right. \\ &\quad \left. + g^{\mu\nu} \square h^\rho_\rho + (\nabla^\mu h_\rho^\nu + \nabla^\nu h_\rho^\mu - \nabla_\rho h^{\mu\nu} - 2g^{\mu\nu} \nabla^\sigma h_{\rho\sigma} + g^{\mu\nu} \nabla_\rho h^\sigma_\sigma) \nabla^\rho - g^{\mu\nu} h_{\rho\sigma} \nabla^\rho \nabla^\sigma \right]. \end{aligned} \quad (2.42)$$

It is understood that indices are raised with the background inverse metric  $g^{\mu\nu}$  and that all the covariant derivatives are associated to the metric  $g_{\mu\nu}$ .

Substituting (2.39) into the  $n$ -dimensional version of the Einstein-Langevin Eq. (2.15), taking into account that  $g_{\mu\nu}$  satisfies the semiclassical Einstein equation (2.7), and substituting expression (2.41) we can write the Einstein-Langevin equation in dimensional regularization as

$$\begin{aligned} & \frac{1}{8\pi G_B} \left[ G^{(1)\mu\nu} - \frac{1}{2} g^{\mu\nu} G^{\rho\sigma} h_{\rho\sigma} + G^{\mu\rho} h_\rho^\nu + G^{\nu\rho} h_\rho^\mu + \Lambda_B \left( h^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h_\rho^\rho \right) \right] (x) \\ & - \frac{4}{3} \alpha_B \left( D^{(1)\mu\nu} - \frac{1}{2} g^{\mu\nu} D^{\rho\sigma} h_{\rho\sigma} + D^{\mu\rho} h_\rho^\nu + D^{\nu\rho} h_\rho^\mu \right) (x) - 2\beta_B \left( B^{(1)\mu\nu} - \frac{1}{2} g^{\mu\nu} B^{\rho\sigma} h_{\rho\sigma} + B^{\mu\rho} h_\rho^\nu + B^{\nu\rho} h_\rho^\mu \right) (x) \\ & - \mu^{-(n-4)} \mathcal{F}_x^{\mu\nu} \langle \hat{\phi}_n^2(x) \rangle [g] + 2 \int d^n y \sqrt{-g(y)} \mu^{-(n-4)} H_n^{\mu\nu\rho\sigma} [g](x, y) h_{\rho\sigma}(y) = 2\mu^{-(n-4)} \xi_n^{\mu\nu}(x), \end{aligned} \quad (2.43)$$

where the tensors  $G^{\mu\nu}$ ,  $D^{\mu\nu}$  and  $B^{\mu\nu}$  are computed from the semiclassical metric  $g_{\mu\nu}$ , and where we have omitted the functional dependence on  $g_{\mu\nu}$  and  $h_{\mu\nu}$  in  $G^{(1)\mu\nu}$ ,  $D^{(1)\mu\nu}$ ,  $B^{(1)\mu\nu}$  and  $\mathcal{F}^{\mu\nu}$  to simplify the notation. The parameter  $\mu$  is a mass scale which relates the dimensions of the physical field  $\phi$  with the dimensions of the corresponding field in  $n$ -dimensions,  $\phi_n = \mu^{(n-4)/2} \phi$ . Notice that, in Eq. (2.43), all the ultraviolet divergencies in the limit  $n \rightarrow 4$ , which must be removed by renormalization of the coupling constants, are in  $\langle \hat{\phi}_n^2(x) \rangle$  and the symmetric part  $H_{S_n}^{\mu\nu\rho\sigma}(x, y)$  of the kernel  $H_n^{\mu\nu\rho\sigma}(x, y)$ , whereas the kernels  $N_n^{\mu\nu\rho\sigma}(x, y)$  and  $H_{A_n}^{\mu\nu\rho\sigma}(x, y)$  are free of ultraviolet divergencies. If we introduce the bi-tensor  $F_n^{\mu\nu\rho\sigma}[g](x, y)$  defined by

$$F_n^{\mu\nu\rho\sigma}[g](x, y) \equiv \langle \hat{t}_n^{\mu\nu}(x) \hat{t}_n^{\rho\sigma}(y) \rangle [g] \quad (2.44)$$

where  $\hat{t}^{\mu\nu}$  is defined by Eq. (2.12), then the kernels  $N$  and  $H_A$  can be written as

$$N_n^{\mu\nu\rho\sigma}[g](x, y) = \frac{1}{4} \text{Re} F_n^{\mu\nu\rho\sigma}[g](x, y), \quad H_{A_n}^{\mu\nu\rho\sigma}[g](x, y) = \frac{1}{4} \text{Im} F_n^{\mu\nu\rho\sigma}[g](x, y), \quad (2.45)$$

where we have used that  $2 \langle \hat{t}^{\mu\nu}(x) \hat{t}^{\rho\sigma}(y) \rangle = \langle \{ \hat{t}^{\mu\nu}(x), \hat{t}^{\rho\sigma}(y) \} \rangle + \langle [ \hat{t}^{\mu\nu}(x), \hat{t}^{\rho\sigma}(y) ] \rangle$ , and the fact that the first term on the right hand side of this identity is real, whereas the second one is pure imaginary. Once we perform the renormalization procedure in Eq. (2.43), setting  $n = 4$  will yield the physical semiclassical Einstein-Langevin equation. Due to the presence of the kernel  $H_n^{\mu\nu\rho\sigma}(x, y)$ , this equation will be usually non-local in the metric perturbation.

## 2. The kernels for the vacuum state

When the expectation values in the Einstein-Langevin equation are taken in a vacuum state  $|0\rangle$ , such as, for instance, an “in” vacuum, we can go further. Since then we can write these expectation values in terms of the Wightman and Feynman functions, defined as

$$G_n^+(x, y) \equiv \langle 0 | \hat{\phi}_n(x) \hat{\phi}_n(y) | 0 \rangle [g], \quad iG_{Fn}(x, y) \equiv \langle 0 | T \left( \hat{\phi}_n(x) \hat{\phi}_n(y) \right) | 0 \rangle [g]. \quad (2.46)$$

These expressions for the kernels in the Einstein-Langevin equation will be very useful for explicit calculations. To simplify the notation, we omit the functional dependence on the semiclassical metric  $g_{\mu\nu}$ , which will be understood in all the expressions below.

From Eqs. (2.45), we see that the kernels  $N_n^{\mu\nu\rho\sigma}(x, y)$  and  $H_{A_n}^{\mu\nu\rho\sigma}(x, y)$  are the real and imaginary parts, respectively, of the bi-tensor  $F_n^{\mu\nu\rho\sigma}(x, y)$ . From the expression (2.4) we see that the stress-energy operator  $\hat{T}_n^{\mu\nu}$  can be written as a sum of terms of the form  $\{ \mathcal{A}_x \hat{\phi}_n(x), \mathcal{B}_x \hat{\phi}_n(x) \}$ , where  $\mathcal{A}_x$  and  $\mathcal{B}_x$  are some differential operators. It then follows that we can express the bi-tensor  $F_n^{\mu\nu\rho\sigma}(x, y)$  in terms of the Wightman function as

$$\begin{aligned} F_n^{\mu\nu\rho\sigma}(x, y) &= \nabla_x^\mu \nabla_y^\rho G_n^+(x, y) \nabla_x^\nu \nabla_y^\sigma G_n^+(x, y) + \nabla_x^\mu \nabla_y^\sigma G_n^+(x, y) \nabla_x^\nu \nabla_y^\rho G_n^+(x, y) \\ &+ 2 \mathcal{D}_x^{\mu\nu} (\nabla_y^\rho G_n^+(x, y) \nabla_y^\sigma G_n^+(x, y)) + 2 \mathcal{D}_y^{\rho\sigma} (\nabla_x^\mu G_n^+(x, y) \nabla_x^\nu G_n^+(x, y)) + 2 \mathcal{D}_x^{\mu\nu} \mathcal{D}_y^{\rho\sigma} (G_n^{+2}(x, y)), \end{aligned} \quad (2.47)$$

where  $\mathcal{D}_x^{\mu\nu}$  is the differential operator (2.5). From this expression and the relations (2.45), we get expressions for the kernels  $N_n$  and  $H_{A_n}$  in terms of the Wightman function  $G_n^+(x, y)$ .

Similarly the kernel  $H_{S_n}^{\mu\nu\rho\sigma}(x, y)$ , can be written in terms of the Feynman function as

$$\begin{aligned}
H_{S_n}^{\mu\nu\rho\sigma}(x, y) = & -\frac{1}{4} \text{Im} \left[ \nabla_x^\mu \nabla_y^\rho G_{F_n}(x, y) \nabla_x^\nu \nabla_y^\sigma G_{F_n}(x, y) + \nabla_x^\mu \nabla_y^\sigma G_{F_n}(x, y) \nabla_x^\nu \nabla_y^\rho G_{F_n}(x, y) \right. \\
& - g^{\mu\nu}(x) \nabla_x^\rho \nabla_y^\sigma G_{F_n}(x, y) \nabla_x^\tau \nabla_y^\sigma G_{F_n}(x, y) - g^{\rho\sigma}(y) \nabla_x^\mu \nabla_y^\tau G_{F_n}(x, y) \nabla_x^\nu \nabla_y^\tau G_{F_n}(x, y) \\
& + \frac{1}{2} g^{\mu\nu}(x) g^{\rho\sigma}(y) \nabla_x^\tau \nabla_y^\kappa G_{F_n}(x, y) \nabla_x^\tau \nabla_y^\kappa G_{F_n}(x, y) + \mathcal{K}_x^{\mu\nu} (2 \nabla_y^\rho G_{F_n}(x, y) \nabla_y^\sigma G_{F_n}(x, y) \\
& - g^{\rho\sigma}(y) \nabla_y^\tau G_{F_n}(x, y) \nabla_y^\tau G_{F_n}(x, y)) + \mathcal{K}_y^{\rho\sigma} (2 \nabla_x^\mu G_{F_n}(x, y) \nabla_x^\nu G_{F_n}(x, y) \\
& \left. - g^{\mu\nu}(x) \nabla_x^\tau G_{F_n}(x, y) \nabla_x^\tau G_{F_n}(x, y)) + 2 \mathcal{K}_x^{\mu\nu} \mathcal{K}_y^{\rho\sigma} (G_{F_n}^2(x, y)) \right], \tag{2.48}
\end{aligned}$$

where  $\mathcal{K}_x^{\mu\nu}$  is the differential operator

$$\mathcal{K}_x^{\mu\nu} \equiv \xi (g^{\mu\nu}(x) \square_x - \nabla_x^\mu \nabla_x^\nu + G^{\mu\nu}(x)) - \frac{1}{2} m^2 g^{\mu\nu}(x). \tag{2.49}$$

Note that, in the vacuum state  $|0\rangle$ , the term  $\langle \hat{\phi}_n^2(x) \rangle$  in equation (2.43) can also be written as  $\langle \hat{\phi}_n^2(x) \rangle = i G_{F_n}(x, x) = G_n^+(x, x)$ .

Finally, the causality of the Einstein-Langevin equation (2.43) can be explicitly seen as follows. The non-local terms in that equation are due to the kernel  $H(x, y)$  which is defined in Eq. (2.28) as the sum of  $H_S(x, y)$  and  $H_A(x, y)$ . Now, when the points  $x$  and  $y$  are spacelike separated,  $\hat{\phi}_n(x)$  and  $\hat{\phi}_n(y)$  commute and, thus,  $G_n^+(x, y) = i G_{F_n}(x, y) = (1/2) \langle 0 | \{ \hat{\phi}_n(x), \hat{\phi}_n(y) \} | 0 \rangle$ , which is real. Hence, from the above expressions, we have that  $H_{A_n}^{\mu\nu\rho\sigma}(x, y) = H_{S_n}^{\mu\nu\rho\sigma}(x, y) = 0$ , and thus  $H_n^{\mu\nu\rho\sigma}(x, y) = 0$ . This fact is not surprising since, from the causality of the expectation value of the stress-energy operator, we know that the non-local dependence on the metric perturbation in the Einstein-Langevin equation, see Eq. (2.14), must be causal.

### 3. Discussion

In this section, based on Refs. [4,5] we have shown how a consistent stochastic semiclassical theory of gravity can be formulated. This theory is a perturbative generalization of semiclassical gravity which describes the back reaction of the lowest order stress-energy fluctuations of quantum matter fields on the gravitational field through the semiclassical Einstein-Langevin equation. We have shown that this equation can be formally derived with a method based on the influence functional of Feynman and Vernon, where one considers the metric field as the “system” of interest and the matter fields as part of its “environment”. An explicit linear form of the Einstein-Langevin equations has been given in terms of some kernels which depend on the Wightman and Feynman functions when a vacuum state is considered.

In Ref. [5] the fluctuation-dissipation relations and particle creation in the stochastic gravity context was also discussed. When the background solution of semiclassical gravity consists of a stationary spacetime and a scalar field in a thermal equilibrium state, it is possible to identify a dissipation kernel in the Einstein-Langevin equation which is related to the noise kernel by a fluctuation-dissipation relation [83,84]. The same relation was previously derived by Mottola [46] using a linear response theory approach, and it is in agreement with previous findings on a Minkowski background [74]; see, however, comments in [47]. This result was also generalized to the case of a conformal scalar field in a conformally stationary background solution of semiclassical gravity.

Particle creation by stochastic metric perturbations in stationary and conformally stationary background solutions of semiclassical gravity was also considered. It is possible to express the total probability of particle creation and the number of created particles (the expectation value of the number operator for “out” particles in the “in” vacuum) in terms of the vacuum noise kernel. Remarkably the averaged value of those quantities is enhanced, over the semiclassical result, by the presence of stochastic metric fluctuations. In the particular cases of a Minkowski background and a conformal field in a spatially flat RW background, the energy of the created particles can be expressed in terms of the vacuum dissipation kernels; see also Refs. [31,85].

## III. METRIC FLUCTUATIONS IN MINKOWSKI SPACETIME

In this section we describe the first application of the full stochastic semiclassical theory of gravity, where we evaluate the stochastic gravitational fluctuations in a Minkowski background. In order to do so, we first use the method developed in section II to derive the semiclassical Einstein-Langevin equation around a class of solutions of semiclassical gravity consisting of Minkowski spacetime and a linear real scalar field in its vacuum state, which may be considered the ground state of semiclassical gravity. Although the Minkowski vacuum is an eigenstate of the

total four-momentum operator of a field in Minkowski spacetime, it is not an eigenstate of the stress-energy operator. Hence, even for these solutions of semiclassical gravity, for which the expectation value of the stress-energy operator can always be chosen to be zero, the fluctuations of this operator are non-vanishing. This fact leads to consider the stochastic corrections to these solutions described by the semiclassical Einstein-Langevin equation.

We then solve the Einstein-Langevin equation for the linearized Einstein tensor and compute the associated two-point correlation functions. Even though, in this case, we expect to have negligibly small values for these correlation functions for points separated by lengths larger than the Planck length, there are several reasons why it is worth carrying out this calculation.

On the one hand, these are the first back-reaction solutions of the full semiclassical Einstein-Langevin equation. There are analogous solutions to a “reduced” version of this equation inspired in a “mini-superspace” model [86], and there is also a previous attempt to obtain a solution to the Einstein-Langevin equation in Ref. [73], but, there, the non-local terms in the Einstein-Langevin equation were neglected.

On the other hand, the results of this calculation, which confirm our expectations that gravitational fluctuations are negligible at length scales larger than the Planck length, but also predict that the fluctuations are strongly suppressed on small scales, can be considered a first test of stochastic semiclassical gravity. In addition, we can extract conclusions on the possible qualitative behavior of the solutions to the Einstein-Langevin equation. Thus, it is interesting to note that the correlation functions at short scales are characterized by correlation lengths of the order of the Planck length; furthermore, such correlation lengths enter in a non-analytic way in the correlation functions. This kind of non-analytic behavior is actually quite common in the solutions to Langevin-type equations with dissipative terms and hints at the possibility that correlation functions for other solutions to the Einstein-Langevin equation are also non-analytic in their characteristic correlation lengths.

### A. Perturbations around Minkowski spacetime

The Minkowski metric  $\eta_{\mu\nu}$  in a manifold  $\mathcal{M}$  which is topologically  $\mathbb{R}^4$  and the usual Minkowski vacuum, denoted as  $|0\rangle$ , are the class of simplest solutions to the semiclassical Einstein equation (2.7), the so called trivial solutions of semiclassical gravity [87]. Note that each possible value of the parameters  $(m^2, \xi)$  leads to a different solution. In fact, we can always choose a renormalization scheme in which the renormalized expectation value  $\langle 0 | \hat{T}_R^{\mu\nu} | 0 \rangle [\eta] = 0$ . Thus, Minkowski spacetime  $(\mathbb{R}^4, \eta_{\mu\nu})$  and the vacuum state  $|0\rangle$  are a solution to the semiclassical Einstein equation with renormalized cosmological constant  $\Lambda = 0$ . The fact that the vacuum expectation value of the renormalized stress-energy operator in Minkowski spacetime should vanish was originally proposed by Wald [57] and it may be understood as a renormalization convention [21, 23]. There are other possible renormalization prescriptions in which such vacuum expectation value is proportional to  $\eta^{\mu\nu}$ , and this would determine the value of the cosmological constant  $\Lambda$  in the semiclassical equation. Of course, all these renormalization schemes give physically equivalent results: the total effective cosmological constant, *i.e.*, the constant of proportionality in the sum of all the terms proportional to the metric in the semiclassical Einstein and Einstein-Langevin equations, has to be zero.

As we have already mentioned the vacuum  $|0\rangle$  is an eigenstate of the total four-momentum operator in Minkowski spacetime, but not an eigenstate of  $\hat{T}_{\mu\nu}^R[\eta]$ . Hence, even in the Minkowski background, there are quantum fluctuations in the stress-energy tensor and, as a result, the noise kernel does not vanish. This fact leads to consider the stochastic corrections to this class of trivial solutions of semiclassical gravity. Since, in this case, the Wightman and Feynman functions (2.46), their values in the two-point coincidence limit, and the products of derivatives of two of such functions appearing in expressions (2.47) and (2.48) are known in dimensional regularization, we can compute the semiclassical Einstein-Langevin equation using the method outlined in section II.

In Minkowski spacetime, the components of the classical stress-energy tensor (2.3) reduce to

$$T^{\mu\nu}[\eta, \phi] = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial^\rho \phi \partial_\rho \phi - \frac{1}{2} \eta^{\mu\nu} m^2 \phi^2 + \xi (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu) \phi^2, \quad (3.1)$$

where  $\square \equiv \partial_\mu \partial^\mu$ , and the formal expression for the components of the corresponding “operator” in dimensional regularization, see Eq. (2.4), is

$$\hat{T}_n^{\mu\nu}[\eta] = \frac{1}{2} \{ \partial^\mu \hat{\phi}_n, \partial^\nu \hat{\phi}_n \} + \mathcal{D}^{\mu\nu} \hat{\phi}_n^2, \quad (3.2)$$

where  $\mathcal{D}^{\mu\nu}$  is the differential operator (2.5), with  $g_{\mu\nu} = \eta_{\mu\nu}$ ,  $R_{\mu\nu} = 0$ , and  $\nabla_\mu = \partial_\mu$ . The field  $\hat{\phi}_n(x)$  is the field operator in the Heisenberg representation in a  $n$ -dimensional Minkowski spacetime, which satisfies the Klein-Gordon equation (2.2). We use here a stress-energy tensor which differs from the canonical one, which corresponds to  $\xi = 0$ , both tensors, however, define the same total momentum.

The Wightman and Feynman functions (2.46) when  $g_{\mu\nu} = \eta_{\mu\nu}$ , are well known:

$$G_n^+(x, y) = i\Delta_n^+(x - y), \quad G_{F_n}(x, y) = \Delta_{F_n}(x - y), \quad (3.3)$$

with

$$\begin{aligned} \Delta_n^+(x) &= -2\pi i \int \frac{d^n k}{(2\pi)^n} e^{ikx} \delta(k^2 + m^2) \theta(k^0), \\ \Delta_{F_n}(x) &= - \int \frac{d^n k}{(2\pi)^n} \frac{e^{ikx}}{k^2 + m^2 - i\epsilon}, \quad \epsilon \rightarrow 0^+, \end{aligned} \quad (3.4)$$

where  $k^2 \equiv \eta_{\mu\nu} k^\mu k^\nu$  and  $kx \equiv \eta_{\mu\nu} k^\mu x^\nu$ . Note that the derivatives of these functions satisfy  $\partial_\mu^x \Delta_n^+(x - y) = \partial_\mu \Delta_n^+(x - y)$  and  $\partial_\mu^y \Delta_n^+(x - y) = -\partial_\mu \Delta_n^+(x - y)$ , and similarly for the Feynman propagator  $\Delta_{F_n}(x - y)$ .

To write down the semiclassical Einstein equation (2.7) in  $n$ -dimensions for this case, we need to compute the vacuum expectation value of the stress-energy operator components (3.2). Since, from (3.3), we have that  $\langle 0 | \hat{\phi}_n^2(x) | 0 \rangle = i\Delta_{F_n}(0) = i\Delta_n^+(0)$ , which is a constant (independent of  $x$ ), we have simply

$$\langle 0 | \hat{T}_n^{\mu\nu} | 0 \rangle [\eta] = -i (\partial^\mu \partial^\nu \Delta_{F_n})(0) = -i \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{k^2 + m^2 - i\epsilon} = \frac{\eta^{\mu\nu}}{2} \left( \frac{m^2}{4\pi} \right)^{n/2} \Gamma\left(-\frac{n}{2}\right), \quad (3.5)$$

where the integrals in dimensional regularization have been computed in the standard way (see Ref. [6]) and where  $\Gamma(z)$  is the Euler's gamma function. The semiclassical Einstein equation (2.7) in  $n$ -dimensions before renormalization reduces now to

$$\frac{\Lambda_B}{8\pi G_B} \eta^{\mu\nu} = \mu^{-(n-4)} \langle 0 | \hat{T}_n^{\mu\nu} | 0 \rangle [\eta]. \quad (3.6)$$

This equation, thus, simply sets the value of the bare coupling constant  $\Lambda_B/G_B$ . Note, from (3.5), that in order to have  $\langle 0 | \hat{T}_R^{\mu\nu} | 0 \rangle [\eta] = 0$ , the renormalized and regularized stress-energy tensor “operator” for a scalar field in Minkowski spacetime, see Eq. (2.6), has to be defined as

$$\hat{T}_R^{\mu\nu}[\eta] = \mu^{-(n-4)} \hat{T}_n^{\mu\nu}[\eta] - \frac{\eta^{\mu\nu}}{2} \frac{m^4}{(4\pi)^2} \left( \frac{m^2}{4\pi\mu^2} \right)^{\frac{n-4}{2}} \Gamma\left(-\frac{n}{2}\right), \quad (3.7)$$

which corresponds to a renormalization of the cosmological constant

$$\frac{\Lambda_B}{G_B} = \frac{\Lambda}{G} - \frac{2}{\pi} \frac{m^4}{n(n-2)} \kappa_n + O(n-4), \quad (3.8)$$

where

$$\kappa_n \equiv \frac{1}{(n-4)} \left( \frac{e^\gamma m^2}{4\pi\mu^2} \right)^{\frac{n-4}{2}} = \frac{1}{n-4} + \frac{1}{2} \ln \left( \frac{e^\gamma m^2}{4\pi\mu^2} \right) + O(n-4), \quad (3.9)$$

being  $\gamma$  the Euler's constant. In the case of a massless scalar field,  $m^2 = 0$ , one simply has  $\Lambda_B/G_B = \Lambda/G$ . Introducing this renormalized coupling constant into Eq. (3.6), we can take the limit  $n \rightarrow 4$ . We find again that, for  $(\mathbb{R}^4, \eta_{ab}, |0\rangle)$  to satisfy the semiclassical Einstein equation, we must take  $\Lambda = 0$ .

We can now write down the Einstein-Langevin equations for the components  $h_{\mu\nu}$  of the stochastic metric perturbation in dimensional regularization. In our case, using  $\langle 0 | \hat{\phi}_n^2(x) | 0 \rangle = i\Delta_{F_n}(0)$  and the explicit expression of Eq. (2.43) we obtain

$$\begin{aligned} & \frac{1}{8\pi G_B} \left[ G^{(1)\mu\nu} + \Lambda_B \left( h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) \right] (x) - \frac{4}{3} \alpha_B D^{(1)\mu\nu}(x) - 2\beta_B B^{(1)\mu\nu}(x) \\ & - \xi G^{(1)\mu\nu}(x) \mu^{-(n-4)} i\Delta_{F_n}(0) + 2 \int d^n y \mu^{-(n-4)} H_n^{\mu\nu\alpha\beta}(x, y) h_{\alpha\beta}(y) = 2\xi^{\mu\nu}(x). \end{aligned} \quad (3.10)$$

The indices in  $h_{\mu\nu}$  are raised with the Minkowski metric and  $h \equiv h^\rho_\rho$ , and here a superindex  $(1)$  denotes the components of a tensor linearized around the flat metric.

Note that in  $n$ -dimensions the correlator for the field  $\xi^{\mu\nu}$  is written as



$$\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(y) \rangle_s = \mu^{-2(n-4)} N_n^{\mu\nu\alpha\beta}(x, y), \quad (3.11)$$

Explicit expressions for  $D^{(1)\mu\nu}$  and  $B^{(1)\mu\nu}$  are given by

$$D^{(1)\mu\nu}(x) = \frac{1}{2} (3\mathcal{F}_x^{\mu\alpha} \mathcal{F}_x^{\nu\beta} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}) h_{\alpha\beta}(x), \quad B^{(1)\mu\nu}(x) = 2\mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} h_{\alpha\beta}(x), \quad (3.12)$$

where  $\mathcal{F}_x^{\mu\nu}$  is the differential operator  $\mathcal{F}_x^{\mu\nu} \equiv \eta^{\mu\nu} \square_x - \partial_x^\mu \partial_x^\nu$ .

## B. The kernels in the Minkowski background

### 1. The noise and dissipation kernels

Since the two kernels (2.45) are free of ultraviolet divergencies in the limit  $n \rightarrow 4$ , we can deal directly with the  $F^{\mu\nu\alpha\beta}(x-y) \equiv \lim_{n \rightarrow 4} \mu^{-2(n-4)} F_n^{\mu\nu\alpha\beta}$  in Eq. (2.44). The kernels  $4N^{\mu\nu\alpha\beta}(x, y) = \text{Re } F^{\mu\nu\alpha\beta}(x-y)$  and  $4H_A^{\mu\nu\alpha\beta}(x, y) = \text{Im } F^{\mu\nu\alpha\beta}(x-y)$  are actually the components of the “physical” noise and dissipation kernels that will appear in the Einstein-Langevin equations once the renormalization procedure has been carried out. The bi-tensor  $F^{\mu\nu\alpha\beta}$  can be expressed in terms of the Wightman function in four spacetime dimensions, in the following way:

$$F^{\mu\nu\alpha\beta}(x) = -2 \left[ \partial^\mu \partial^{(\alpha} \Delta^+(x) \partial^{\beta)} \partial^\nu \Delta^+(x) + \mathcal{D}^{\mu\nu} (\partial^\alpha \Delta^+(x) \partial^\beta \Delta^+(x)) \right. \\ \left. + \mathcal{D}^{\alpha\beta} (\partial^\mu \Delta^+(x) \partial^\nu \Delta^+(x)) + \mathcal{D}^{\mu\nu} \mathcal{D}^{\alpha\beta} (\Delta^{+2}(x)) \right]. \quad (3.13)$$

The different terms in Eq. (3.13) can be easily computed using the integrals

$$I(p) \equiv \int \frac{d^4 k}{(2\pi)^4} \delta(k^2 + m^2) \theta(-k^0) \delta[(k-p)^2 + m^2] \theta(k^0 - p^0), \quad (3.14)$$

and  $I^{\mu_1 \dots \mu_r}(p)$  which are defined as the previous one by inserting the momenta  $k^{\mu_1} \dots k^{\mu_r}$  with  $r = 1, 2, 3, 4$  in the integrand. All these integral can be expressed in terms of  $I(p)$ ; see Ref. [6] for the explicit expressions. It is convenient to separate  $I(p)$  in its even and odd parts with respect to the variables  $p^\mu$  as

$$I(p) = I_S(p) + I_A(p), \quad (3.15)$$

where  $I_S(-p) = I_S(p)$  and  $I_A(-p) = -I_A(p)$ . These two functions are explicitly given by

$$I_S(p) = \frac{1}{8(2\pi)^3} \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}}, \\ I_A(p) = \frac{-1}{8(2\pi)^3} \text{sign } p^0 \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}}. \quad (3.16)$$

After some manipulations, we find

$$F^{\mu\nu\alpha\beta}(x) = \frac{\pi^2}{45} (3\mathcal{F}_x^{\mu(\alpha} \mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}) \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \left(1 + 4 \frac{m^2}{p^2}\right)^2 I(p) \\ + \frac{8\pi^2}{9} \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 I(p), \quad (3.17)$$

where  $\Delta\xi \equiv \xi - 1/6$ . The real and imaginary parts of the last expression, which yield the noise and dissipation kernels, are easily recognized as the terms containing  $I_S(p)$  and  $I_A(p)$ , respectively. To write them explicitly, it is useful to introduce the new kernels

$$N_A(x; m^2) \equiv \frac{1}{1920\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left(1 + 4 \frac{m^2}{p^2}\right)^2, \\ N_B(x; m^2, \Delta\xi) \equiv \frac{1}{288\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left(3\Delta\xi + \frac{m^2}{p^2}\right)^2,$$

$$\begin{aligned}
D_A(x; m^2) &\equiv \frac{-i}{1920\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \text{sign } p^0 \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left(1 + 4 \frac{m^2}{p^2}\right)^2, \\
D_B(x; m^2, \Delta\xi) &\equiv \frac{-i}{288\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \text{sign } p^0 \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left(3\Delta\xi + \frac{m^2}{p^2}\right)^2,
\end{aligned} \tag{3.18}$$

and we finally get

$$\begin{aligned}
N^{\mu\nu\alpha\beta}(x, y) &= \frac{1}{6} (3\mathcal{F}_x^{\mu(\alpha} \mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}) N_A(x-y; m^2) + \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} N_B(x-y; m^2, \Delta\xi), \\
H_A^{\mu\nu\alpha\beta}(x, y) &= \frac{1}{6} (3\mathcal{F}_x^{\mu(\alpha} \mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}) D_A(x-y; m^2) + \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} D_B(x-y; m^2, \Delta\xi).
\end{aligned} \tag{3.19}$$

Notice that the noise and dissipation kernels defined in (3.18) are actually real because, for the noise kernels, only the  $\cos px$  terms of the exponentials  $e^{ipx}$  contribute to the integrals, and, for the dissipation kernels, the only contribution of such exponentials comes from the  $i \sin px$  terms.

## 2. The kernel $H_S^{\mu\nu\alpha\beta}(x, y)$

The evaluation of the kernel components  $H_{S_n}^{\mu\nu\alpha\beta}(x, y)$  is a much more cumbersome task. Since these quantities contain divergencies in the limit  $n \rightarrow 4$ , we shall compute them using dimensional regularization. Using Eq. (2.48), these components can be written in terms of the Feynman propagator (3.4) as

$$\mu^{-(n-4)} H_{S_n}^{\mu\nu\alpha\beta}(x, y) = \frac{1}{4} \text{Im } K^{\mu\nu\alpha\beta}(x - y), \tag{3.20}$$

where

$$\begin{aligned}
K^{\mu\nu\alpha\beta}(x) &\equiv -\mu^{-(n-4)} \left\{ 2\partial^\mu \partial^{(\alpha} \Delta_{F_n}(x) \partial^{\beta)} \partial^\nu \Delta_{F_n}(x) + 2\mathcal{D}^{\mu\nu} (\partial^\alpha \Delta_{F_n}(x) \partial^\beta \Delta_{F_n}(x)) \right. \\
&\quad + 2\mathcal{D}^{\alpha\beta} (\partial^\mu \Delta_{F_n}(x) \partial^\nu \Delta_{F_n}(x)) + 2\mathcal{D}^{\mu\nu} \mathcal{D}^{\alpha\beta} (\Delta_{F_n}^2(x)) + \left[ \eta^{\mu\nu} \partial^{(\alpha} \Delta_{F_n}(x) \partial^{\beta)} + \eta^{\alpha\beta} \partial^{(\mu} \Delta_{F_n}(x) \partial^{\nu)} \right. \\
&\quad \left. \left. + \Delta_{F_n}(0) (\eta^{\mu\nu} \mathcal{D}^{\alpha\beta} + \eta^{\alpha\beta} \mathcal{D}^{\mu\nu}) + \frac{1}{4} \eta^{\mu\nu} \eta^{\alpha\beta} (\Delta_{F_n}(x) \square - m^2 \Delta_{F_n}(0)) \right] \delta^n(x) \right\}.
\end{aligned} \tag{3.21}$$

Let us define the integrals

$$J_n(p) \equiv \mu^{-(n-4)} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + m^2 - i\epsilon) [(k-p)^2 + m^2 - i\epsilon]}, \tag{3.22}$$

and  $J_n^{\mu_1 \dots \mu_r}(p)$  obtained by inserting the momenta  $k^{\mu_1} \dots k^{\mu_r}$  into the previous integral, together with

$$I_{0n} \equiv \mu^{-(n-4)} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + m^2 - i\epsilon)}, \tag{3.23}$$

and  $I_{0n}^{\mu_1 \dots \mu_r}$  which are also obtained by inserting momenta in the integrand. Then, the different terms in Eq. (3.21) can be computed. These integrals are explicitly given in Ref. [6]. It is found that  $I_{0n}^\mu = 0$  and the remaining integrals can be written in terms of  $I_{0n}$  and  $J_n(p)$ . It is useful to introduce the projector  $P^{\mu\nu}$  orthogonal to  $p^\mu$  as

$$p^2 P^{\mu\nu} \equiv \eta^{\mu\nu} p^2 - p^\mu p^\nu \tag{3.24}$$

then the action of the operator  $\mathcal{F}_x^{\mu\nu}$  is simply written as  $\mathcal{F}_x^{\mu\nu} \int d^n p e^{ipx} f(p) = -\int d^n p e^{ipx} f(p) p^2 P^{\mu\nu}$  where  $f(p)$  is an arbitrary function of  $p^\mu$ .

After a rather long but straightforward calculation, we get, expanding around  $n=4$ ,

$$\begin{aligned}
K^{\mu\nu\alpha\beta}(x) = & \frac{i}{(4\pi)^2} \left\{ \kappa_n \left[ \frac{1}{90} (3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}) \delta^n(x) + 4\Delta\xi^2 \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta} \delta^n(x) \right. \right. \\
& + \frac{2}{3} \frac{m^2}{(n-2)} (\eta^{\mu\nu}\eta^{\alpha\beta}\square_x - \eta^{\mu(\alpha}\eta^{\beta)\nu}\square_x + \eta^{\mu(\alpha}\partial_x^{\beta)}\partial_x^\nu + \eta^{\nu(\alpha}\partial_x^{\beta)}\partial_x^\mu - \eta^{\mu\nu}\partial_x^\alpha\partial_x^\beta - \eta^{\alpha\beta}\partial_x^\mu\partial_x^\nu) \delta^n(x) \\
& + \frac{4m^4}{n(n-2)} (2\eta^{\mu(\alpha}\eta^{\beta)\nu} - \eta^{\mu\nu}\eta^{\alpha\beta}) \delta^n(x) \left. \right] + \frac{1}{180} (3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}) \\
& \times \int \frac{d^n p}{(2\pi)^n} e^{ipx} \left( 1 + 4 \frac{m^2}{p^2} \right)^2 \bar{\phi}(p^2) + \frac{2}{9} \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta} \int \frac{d^n p}{(2\pi)^n} e^{ipx} \left( 3\Delta\xi + \frac{m^2}{p^2} \right)^2 \bar{\phi}(p^2) \\
& - \left[ \frac{4}{675} (3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}) + \frac{1}{270} (60\xi - 11) \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta} \right] \delta^n(x) \\
& - m^2 \left[ \frac{2}{135} (3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}) + \frac{1}{27} \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta} \right] \Delta_n(x) \left. \right\} + O(n-4), \tag{3.25}
\end{aligned}$$

where  $\kappa_n$  has been defined in (3.9), and  $\bar{\phi}(p^2)$  and  $\Delta_n(x)$  are given by

$$\bar{\phi}(p^2) \equiv \int_0^1 d\alpha \ln \left( 1 + \frac{p^2}{m^2} \alpha(1-\alpha) - i\epsilon \right) = -i\pi \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} + \varphi(p^2), \tag{3.26}$$

$$\Delta_n(x) \equiv \int \frac{d^n p}{(2\pi)^n} e^{ipx} \frac{1}{p^2}, \tag{3.27}$$

where  $\varphi(p^2) \equiv \int_0^1 d\alpha \ln |1 + \frac{p^2}{m^2} \alpha(1-\alpha)|$ .

The imaginary part of (3.25) [which, using (3.20), gives the kernel components  $\mu^{-(n-4)} H_{s_n}^{\mu\nu\alpha\beta}(x, y)$ ] can be easily obtained multiplying this expression by  $-i$  and retaining only the real part,  $\varphi(p^2)$ , of the function  $\bar{\phi}(p^2)$ . Making use of this result, it is easy to compute the contribution of these kernel components to the Einstein-Langevin equation.

### C. The Einstein-Langevin equations

With the results of the previous subsections we can now write the  $n$ -dimensional Einstein-Langevin equation (3.10), previous to the renormalization. Taking into account that, from Eqs. (3.5) and (3.6),

$$\frac{\Lambda_B}{8\pi G_B} = -\frac{1}{4\pi^2} \frac{m^4}{n(n-2)} \kappa_n + O(n-4), \tag{3.28}$$

we may finally write:

$$\begin{aligned}
& \frac{1}{8\pi G_B} G^{(1)\mu\nu}(x) - \frac{4}{3} \alpha_B D^{(1)\mu\nu}(x) - 2\beta_B B^{(1)\mu\nu}(x) + \frac{\kappa_n}{(4\pi)^2} \left[ -4\Delta\xi \frac{m^2}{(n-2)} G^{(1)\mu\nu} + \frac{1}{90} D^{(1)\mu\nu} \right. \\
& + \Delta\xi^2 B^{(1)\mu\nu} \left. \right](x) + \frac{1}{2880\pi^2} \left\{ -\frac{16}{15} D^{(1)\mu\nu}(x) + \left( \frac{1}{6} - 10\Delta\xi \right) B^{(1)\mu\nu}(x) \right. \\
& + \int d^n y \int \frac{d^n p}{(2\pi)^n} e^{ip(x-y)} \varphi(p^2) \left[ \left( 1 + 4 \frac{m^2}{p^2} \right)^2 D^{(1)\mu\nu}(y) + 10 \left( 3\Delta\xi + \frac{m^2}{p^2} \right)^2 B^{(1)\mu\nu}(y) \right] \\
& - \frac{m^2}{3} \int d^n y \Delta_n(x-y) \left( 8D^{(1)\mu\nu} + 5B^{(1)\mu\nu} \right)(y) \left. \right\} + 2 \int d^n y \mu^{-(n-4)} H_{\Lambda_n}^{\mu\nu\alpha\beta}(x, y) h_{\alpha\beta}(y) + O(n-4) \\
& = 2\xi^{\mu\nu}(x). \tag{3.29}
\end{aligned}$$

Notice that the terms containing the bare cosmological constant have canceled. These equations can now be renormalized, that is, we can now write the bare coupling constants as renormalized coupling constants plus some suitably chosen counterterms and take the limit  $n \rightarrow 4$ . In order to carry out such a procedure, it is convenient to distinguish between massive and massless scalar fields. We shall evaluate these two cases in different subsections.

### 1. Massive field

In the case of a scalar field with mass  $m \neq 0$ , we can use, as we have done in Eq. (3.8) for the cosmological constant, the renormalized coupling constants  $1/G$ ,  $\alpha$  and  $\beta$  as

$$\begin{aligned}\frac{1}{G_B} &= \frac{1}{G} + \frac{2}{\pi} \Delta\xi \frac{m^2}{(n-2)} \kappa_n + O(n-4), \\ \alpha_B &= \alpha + \frac{1}{(4\pi)^2} \frac{1}{120} \kappa_n + O(n-4), \\ \beta_B &= \beta + \frac{\Delta\xi^2}{32\pi^2} \kappa_n + O(n-4).\end{aligned}\tag{3.30}$$

Note that for conformal coupling,  $\Delta\xi = 0$ , one has  $1/G_B = 1/G$  and  $\beta_B = \beta$ , that is, only the coupling constant  $\alpha$  and the cosmological constant need renormalization.

Let us introduce the two new kernels

$$\begin{aligned}H_A(x; m^2) &\equiv \frac{1}{1920\pi^2} \int \frac{d^4p}{(2\pi)^4} e^{ipx} \left\{ \left(1 + 4 \frac{m^2}{p^2}\right)^2 \left[ -i\pi \operatorname{sign} p^0 \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \right. \right. \\ &\quad \left. \left. + \varphi(p^2) \right] - \frac{8}{3} \frac{m^2}{p^2} \right\}, \\ H_B(x; m^2, \Delta\xi) &\equiv \frac{1}{288\pi^2} \int \frac{d^4p}{(2\pi)^4} e^{ipx} \left\{ \left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 \left[ -i\pi \operatorname{sign} p^0 \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \right. \right. \\ &\quad \left. \left. + \varphi(p^2) \right] - \frac{1}{6} \frac{m^2}{p^2} \right\},\end{aligned}\tag{3.31}$$

where  $\varphi(p^2)$  is given by the restriction to  $n = 4$  of expression (3.26). Substituting expressions (3.30) into Eq. (3.29), we can now take the limit  $n \rightarrow 4$ , using the fact that, for  $n = 4$ ,  $D^{(1)\mu\nu}(x) = (3/2) A^{(1)\mu\nu}(x)$ , we obtain the semiclassical Einstein-Langevin equations for the physical stochastic perturbations  $h_{\mu\nu}$  in the four-dimensional manifold  $\mathcal{M} \equiv \mathbb{R}^4$ :

$$\begin{aligned}\frac{1}{8\pi G} G^{(1)\mu\nu}(x) - 2(\alpha A^{(1)\mu\nu}(x) + \beta B^{(1)\mu\nu}(x)) + \frac{1}{2880\pi^2} \left[ -\frac{8}{5} A^{(1)\mu\nu}(x) + \left(\frac{1}{6} - 10\Delta\xi\right) B^{(1)\mu\nu}(x) \right] \\ + \int d^4y [H_A(x-y; m^2) A^{(1)\mu\nu}(y) + H_B(x-y; m^2, \Delta\xi) B^{(1)\mu\nu}(y)] = 2\xi^{\mu\nu}(x),\end{aligned}\tag{3.32}$$

where  $\xi^{\mu\nu}$  are the components of a Gaussian stochastic tensor of vanishing mean value and two-point correlation function  $\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(y) \rangle_s = N^{\mu\nu\alpha\beta}(x, y)$ , given by one of the Eqs. (3.19). Note that the two kernels defined in (3.31) are real and can be split into an even part and an odd part with respect to the variables  $x^\mu$ , with the odd terms being the dissipation kernels  $D_A(x; m^2)$  and  $D_B(x; m^2, \Delta\xi)$  defined in (3.18). In spite of appearances, one can show that the Fourier transforms of the even parts of these kernels are finite in the limit  $p^2 \rightarrow 0$  and, hence, the kernels  $H_A$  and  $H_B$  are well defined distributions.

We should mention that in Ref. [88], the same Einstein-Langevin equations were calculated using rather different methods. The way in which the result is written makes difficult a direct comparison with our equations (3.32). For instance, it is not obvious that there is some analog of the dissipation kernels.

### 2. Massless field

In this subsection, we consider the limit  $m \rightarrow 0$  of equations (3.29). The renormalization scheme used in the previous subsection becomes singular in the massless limit because the expressions (3.30) for  $\alpha_B$  and  $\beta_B$  diverge when  $m \rightarrow 0$ . Therefore, a different renormalization scheme is needed in this case. First, note that we may separate  $\kappa_n$  in (3.9) as  $\kappa_n = \tilde{\kappa}_n + \frac{1}{2} \ln(m^2/\mu^2) + O(n-4)$ , where

$$\tilde{\kappa}_n \equiv \frac{1}{(n-4)} \left( \frac{e^\gamma}{4\pi} \right)^{\frac{n-4}{2}} = \frac{1}{n-4} + \frac{1}{2} \ln \left( \frac{e^\gamma}{4\pi} \right) + O(n-4),\tag{3.33}$$

and that, from Eq. (3.26), we have

$$\lim_{m^2 \rightarrow 0} [\varphi(p^2) + \ln(m^2/\mu^2)] = -2 + \ln \left| \frac{p^2}{\mu^2} \right|. \quad (3.34)$$

Hence, in the massless limit, equations (3.29) reduce to

$$\begin{aligned} & \frac{1}{8\pi G_B} G^{(1)\mu\nu}(x) - \frac{4}{3} \alpha_B D^{(1)\mu\nu}(x) - 2\beta_B B^{(1)\mu\nu}(x) + \frac{1}{(4\pi)^2} (\tilde{\kappa}_n - 1) \left[ \frac{1}{90} D^{(1)\mu\nu} + \Delta \xi^2 B^{(1)\mu\nu} \right](x) \\ & + \frac{1}{2880\pi^2} \left\{ -\frac{16}{15} D^{(1)\mu\nu}(x) + \left( \frac{1}{6} - 10\Delta\xi \right) B^{(1)\mu\nu}(x) + \int d^n y \int \frac{d^n p}{(2\pi)^n} e^{ip(x-y)} \ln \left| \frac{p^2}{\mu^2} \right| \left[ D^{(1)\mu\nu}(y) \right. \right. \\ & \left. \left. + 90\Delta\xi^2 B^{(1)\mu\nu}(y) \right] \right\} + \lim_{m^2 \rightarrow 0} 2 \int d^n y \mu^{-(n-4)} H_{A_n}^{\mu\nu\alpha\beta}(x, y) h_{\alpha\beta}(y) + O(n-4) = 2\xi^{\mu\nu}(x). \end{aligned} \quad (3.35)$$

These equations can be renormalized by introducing the renormalized coupling constants  $1/G$ ,  $\alpha$  and  $\beta$  as

$$\frac{1}{G_B} = \frac{1}{G}, \quad \alpha_B = \alpha + \frac{1}{(4\pi)^2} \frac{1}{120} (\tilde{\kappa}_n - 1) + O(n-4), \quad \beta_B = \beta + \frac{\Delta\xi^2}{32\pi^2} (\tilde{\kappa}_n - 1) + O(n-4). \quad (3.36)$$

Thus, in the massless limit, the Newtonian gravitational constant is not renormalized and, in the conformal coupling case,  $\Delta\xi = 0$ , we have again that  $\beta_B = \beta$ . Introducing the last expressions into Eq. (3.35), we can take the limit  $n \rightarrow 4$ . Note that, by making  $m=0$  in (3.18), the noise and dissipation kernels can be written as

$$\begin{aligned} N_A(x; m^2=0) &= N(x), & N_B(x; m^2=0, \Delta\xi) &= 60\Delta\xi^2 N(x), \\ D_A(x; m^2=0) &= D(x), & D_B(x; m^2=0, \Delta\xi) &= 60\Delta\xi^2 D(x), \end{aligned} \quad (3.37)$$

where

$$N(x) \equiv \frac{1}{1920\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \theta(-p^2), \quad D(x) \equiv \frac{-i}{1920\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \text{sign } p^0 \theta(-p^2). \quad (3.38)$$

It is now convenient to introduce the new kernel

$$\begin{aligned} H(x; \mu^2) &\equiv \frac{1}{1920\pi^2} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \left[ \ln \left| \frac{p^2}{\mu^2} \right| - i\pi \text{sign } p^0 \theta(-p^2) \right] \\ &= \frac{1}{1920\pi^2} \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \ln \left( \frac{-(p^0 + i\epsilon)^2 + p^i p_i}{\mu^2} \right). \end{aligned} \quad (3.39)$$

Again, this kernel is real and can be written as the sum of an even part and an odd part in the variables  $x^\mu$ , where the odd part is the dissipation kernel  $D(x)$ . The Fourier transforms (3.38) and (3.39) can actually be computed and, thus, in this case, we have explicit expressions for the kernels in position space. For  $N(x)$  and  $D(x)$ , we get (see, for instance, Ref. [89])

$$N(x) = \frac{1}{1920\pi} \left[ \frac{1}{\pi^3} \mathcal{P}f \left( \frac{1}{(x^2)^2} \right) + \delta^4(x) \right], \quad D(x) = \frac{1}{1920\pi^3} \text{sign } x^0 \frac{d}{d(x^2)} \delta(x^2), \quad (3.40)$$

where  $\mathcal{P}f$  denotes a distribution generated by the Hadamard finite part of a divergent integral, see Refs. [90] for the definition of these distributions. The expression for the kernel  $H(x; \mu^2)$  can be found in Refs. [91,92] and it is given by

$$\begin{aligned} H(x; \mu^2) &= \frac{1}{960\pi^2} \left\{ \mathcal{P}f \left( \frac{1}{\pi} \theta(x^0) \frac{d}{d(x^2)} \delta(x^2) \right) + (1 - \gamma - \ln \mu) \delta^4(x) \right\} \\ &= \frac{1}{960\pi^2} \lim_{\lambda \rightarrow 0^+} \left\{ \frac{1}{\pi} \theta(x^0) \theta(|\mathbf{x}| - \lambda) \frac{d}{d(x^2)} \delta(x^2) + [1 - \gamma - \ln(\mu\lambda)] \delta^4(x) \right\}. \end{aligned} \quad (3.41)$$

See Ref. [91] for the details on how this last distribution acts on a test function.

Finally, the semiclassical Einstein-Langevin equations for the physical stochastic perturbations  $h_{\mu\nu}$  in the massless case are

$$\begin{aligned} & \frac{1}{8\pi G} G^{(1)\mu\nu}(x) - 2(\alpha A^{(1)\mu\nu}(x) + \beta B^{(1)\mu\nu}(x)) + \frac{1}{2880\pi^2} \left[ -\frac{8}{5} A^{(1)\mu\nu}(x) + \left( \frac{1}{6} - 10\Delta\xi \right) B^{(1)\mu\nu}(x) \right] \\ & + \int d^4y H(x-y; \mu^2) [A^{(1)\mu\nu}(y) + 60\Delta\xi^2 B^{(1)\mu\nu}(y)] = 2\xi^{\mu\nu}(x), \end{aligned} \quad (3.42)$$

where the Gaussian stochastic source components  $\xi^{\mu\nu}$  have zero mean value and

$$\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(y) \rangle_s = \lim_{m \rightarrow 0} N^{\mu\nu\alpha\beta}(x, y) = \left[ \frac{1}{6} (3\mathcal{F}_x^{\mu(\alpha} \mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}) + 60\Delta\xi^2 \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} \right] N(x-y). \quad (3.43)$$

It is interesting to consider the conformally coupled scalar field, *i.e.*, the case  $\Delta\xi=0$ , of particular interest because of its similarities with the electromagnetic field, and also because of its interest in cosmology: massive fields become conformally invariant when their masses are negligible compared to the spacetime curvature. We have already mentioned that for a conformally coupled, field, the stochastic source tensor must be “traceless” (up to first order in perturbation theory around semiclassical gravity), in the sense that the stochastic variable  $\xi_\mu^\mu \equiv \eta_{\mu\nu} \xi^{\mu\nu}$  behaves deterministically as a vanishing scalar field. This can be directly checked by noticing, from Eq. (3.43), that, when  $\Delta\xi=0$ , one has  $\langle \xi_\mu^\mu(x) \xi^{\alpha\beta}(y) \rangle_s = 0$ , since  $\mathcal{F}_\mu^\mu = 3\Box$  and  $\mathcal{F}^{\mu\alpha} \mathcal{F}_\mu^\beta = \Box \mathcal{F}^{\alpha\beta}$ . The semiclassical Einstein-Langevin equations for this particular case (and generalized to a spatially flat Robertson-Walker background) were first obtained in Ref. [73], where the coupling constant  $\beta$  was fixed to be zero.

Note that the expectation value of the renormalized stress-energy tensor for a scalar field can be obtained by identification of Eqs. (3.32) and (3.42) with the components of the physical Einstein-Langevin equation (2.14). The explicit expressions are given in Ref. [6]. The results agree with the general form found by Horowitz [92,93] using an axiomatic approach and coincides with that given in Ref. [87]. The particular cases of conformal coupling,  $\Delta\xi=0$ , and minimal coupling,  $\Delta\xi=-1/6$ , are also in agreement with the results for these cases given in Refs. [92–94,29,95] (modulo local terms proportional to  $A^{(1)\mu\nu}$  and  $B^{(1)\mu\nu}$  due to different choices of the renormalization scheme). For the case of a massive minimally coupled scalar field,  $\Delta\xi=-1/6$ , our result is equivalent to that of Ref. [96].

#### D. Correlation functions for gravitational perturbations

In this section, we solve the semiclassical Einstein-Langevin equations (3.32) and (3.42) for the components  $G^{(1)\mu\nu}$  of the linearized Einstein tensor. In the first subsection we use these solutions to compute the corresponding two-point correlation functions, which give a measure of the gravitational fluctuations predicted by the stochastic semiclassical theory of gravity in the present case. Since the linearized Einstein tensor is invariant under gauge transformations of the metric perturbations, these two-point correlation functions are also gauge invariant. Once we have computed the two-point correlation functions for the linearized Einstein tensor, we find the solutions for the metric perturbations in the next subsection and we show how the associated two-point correlation functions can be computed. This procedure to solve the Einstein-Langevin equations is similar to the one used by Horowitz [92], see also Ref. [87], to analyze the stability of Minkowski spacetime in semiclassical gravity.

We first note that the tensors  $A^{(1)\mu\nu}$  and  $B^{(1)\mu\nu}$  can be written in terms of  $G^{(1)\mu\nu}$  as

$$A^{(1)\mu\nu} = \frac{2}{3} (\mathcal{F}^{\mu\nu} G^{(1)\alpha}_{\alpha} - \mathcal{F}^{\alpha}_{\alpha} G^{(1)\mu\nu}), \quad B^{(1)\mu\nu} = 2\mathcal{F}^{\mu\nu} G^{(1)\alpha}_{\alpha}, \quad (3.44)$$

where we have used that  $3\Box = \mathcal{F}^{\alpha}_{\alpha}$ . Therefore, the Einstein-Langevin equations (3.32) and (3.42) can be seen as linear integro-differential stochastic equations for the components  $G^{(1)\mu\nu}$ . These Einstein-Langevin equations can be written in a unified form, in both cases for  $m \neq 0$  and for  $m=0$ , as

$$\frac{1}{8\pi G} G^{(1)\mu\nu}(x) - 2(\bar{\alpha} A^{(1)\mu\nu}(x) + \bar{\beta} B^{(1)\mu\nu}(x)) + \int d^4y [H_A(x-y) A^{(1)\mu\nu}(y) + H_B(x-y) B^{(1)\mu\nu}(y)] = 2\xi^{\mu\nu}(x), \quad (3.45)$$

where the new constants  $\bar{\alpha}$  and  $\bar{\beta}$ , and the kernels  $H_A(x)$  and  $H_B(x)$  can be identified in each case by comparison of this last equation with Eqs. (3.32) and (3.42). For instance, when  $m=0$ , we have  $H_A(x) = H(x; \mu^2)$  and  $H_B(x) = 60\Delta\xi^2 H(x; \mu^2)$ . In this case, we can use the arbitrariness of the mass scale  $\mu$  to eliminate one of the parameters  $\bar{\alpha}$  or  $\bar{\beta}$ .

In order to find solutions to Eq. (3.45), it is convenient to Fourier transform them. Introducing Fourier transforms with the following convention  $\tilde{f}(p) = \int d^4x e^{-ipx} f(x)$  for a given field  $f(x)$ , one finds, from (3.44),

$$\tilde{A}^{(1)\mu\nu}(p) = 2p^2 \tilde{G}^{(1)\mu\nu}(p) - \frac{2}{3} p^2 P^{\mu\nu} \tilde{G}^{(1)\alpha}_{\alpha}(p), \quad \tilde{B}^{(1)\mu\nu}(p) = -2p^2 P^{\mu\nu} \tilde{G}^{(1)\alpha}_{\alpha}(p). \quad (3.46)$$

Using these relations, the Fourier transform of the Einstein-Langevin Eq. (3.45) reads

$$F^{\mu\nu}_{\alpha\beta}(p) \tilde{G}^{(1)\alpha\beta}(p) = 16\pi G \tilde{\xi}^{\mu\nu}(p), \quad (3.47)$$

where

$$F^{\mu\nu}_{\alpha\beta}(p) \equiv F_1(p) \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu} + F_2(p) p^2 P^{\mu\nu} \eta_{\alpha\beta}, \quad (3.48)$$

with

$$F_1(p) \equiv 1 + 16\pi G p^2 [\tilde{H}_A(p) - 2\bar{\alpha}], \quad F_2(p) \equiv -\frac{16}{3} \pi G [\tilde{H}_A(p) + 3\tilde{H}_B(p) - 2\bar{\alpha} - 6\bar{\beta}]. \quad (3.49)$$

In the Fourier transformed Einstein-Langevin Eq. (3.47),  $\tilde{\xi}^{\mu\nu}(p)$ , the Fourier transform of  $\xi^{\mu\nu}(x)$ , is a Gaussian stochastic source of zero average and

$$\langle \tilde{\xi}^{\mu\nu}(p) \tilde{\xi}^{\alpha\beta}(p') \rangle_s = (2\pi)^4 \delta^4(p + p') \tilde{N}^{\mu\nu\alpha\beta}(p), \quad (3.50)$$

where we have introduced the Fourier transform of the noise kernel. The explicit expression for  $\tilde{N}^{\mu\nu\alpha\beta}(p)$  is found from (3.19) and (3.18) to be

$$\begin{aligned} \tilde{N}^{\mu\nu\alpha\beta}(p) = \frac{1}{2880\pi} \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left[ \frac{1}{4} \left( 1 + 4 \frac{m^2}{p^2} \right)^2 (p^2)^2 (3P^{\mu(\alpha} P^{\beta)\nu} - P^{\mu\nu} P^{\alpha\beta}) \right. \\ \left. + 10 \left( 3\Delta\xi + \frac{m^2}{p^2} \right)^2 (p^2)^2 P^{\mu\nu} P^{\alpha\beta} \right], \end{aligned} \quad (3.51)$$

which in the massless case reduces to

$$\lim_{m \rightarrow 0} \tilde{N}^{\mu\nu\alpha\beta}(p) = \frac{1}{1920\pi} \theta(-p^2) \left[ \frac{1}{6} (p^2)^2 (3P^{\mu(\alpha} P^{\beta)\nu} - P^{\mu\nu} P^{\alpha\beta}) + 60\Delta\xi^2 (p^2)^2 P^{\mu\nu} P^{\alpha\beta} \right]. \quad (3.52)$$

### 1. Correlation functions for the linearized Einstein tensor

In general, we can write  $G^{(1)\mu\nu} = \langle G^{(1)\mu\nu} \rangle_s + G_f^{(1)\mu\nu}$ , where  $G_f^{(1)\mu\nu}$  is a solution to Eqs. (3.45) (or, in the Fourier transformed version, (3.47)) with zero average. The averages  $\langle G^{(1)\mu\nu} \rangle_s$  must be a solution of the linearized semiclassical Einstein equations obtained by averaging Eqs. (3.45) [or (3.47)]. Solutions to these equations (specially in the massless case,  $m = 0$ ) have been studied by several authors [92, 93, 97–99, 95, 87], particularly in connection with the problem of the stability of the ground state of semiclassical gravity. The two-point correlation functions for the linearized Einstein tensor are defined by

$$\mathcal{G}^{\mu\nu\alpha\beta}(x, x') \equiv \langle G^{(1)\mu\nu}(x) G^{(1)\alpha\beta}(x') \rangle_s - \langle G^{(1)\mu\nu}(x) \rangle_s \langle G^{(1)\alpha\beta}(x') \rangle_s = \langle G_f^{(1)\mu\nu}(x) G_f^{(1)\alpha\beta}(x') \rangle_s. \quad (3.53)$$

Now we shall seek the family of solutions to the Einstein-Langevin equations which can be written as a linear functional of the stochastic source and whose Fourier transform,  $\tilde{G}^{(1)\mu\nu}(p)$ , depends locally on  $\tilde{\xi}^{\alpha\beta}(p)$ . Each of such solutions is a Gaussian stochastic field and, thus, it can be completely characterized by the averages  $\langle G^{(1)\mu\nu} \rangle_s$  and the two-point correlation functions (3.53). For such a family of solutions,  $\tilde{G}_f^{(1)\mu\nu}(p)$  is the most general solution to Eq. (3.47) which is linear, homogeneous and local in  $\tilde{\xi}^{\alpha\beta}(p)$ . It can be written as

$$\tilde{G}_f^{(1)\mu\nu}(p) = 16\pi G D^{\mu\nu}_{\alpha\beta}(p) \tilde{\xi}^{\alpha\beta}(p), \quad (3.54)$$

where  $D^{\mu\nu}_{\alpha\beta}(p)$  are the components of a Lorentz invariant tensor field distribution in Minkowski spacetime (by “Lorentz invariant” we mean invariant under the transformations of the orthochronous Lorentz subgroup; see Ref. [92] for more details on the definition and properties of these tensor distributions), symmetric under the interchanges  $\alpha \leftrightarrow \beta$  and  $\mu \leftrightarrow \nu$ , which is the most general solution of

$$F^{\mu\nu}{}_{\rho\sigma}(p) D^{\rho\sigma}{}_{\alpha\beta}(p) = \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu}. \quad (3.55)$$

In addition, we must impose the conservation condition to the solutions:  $p_\nu \tilde{G}_f^{(1)\mu\nu}(p) = 0$ , where this zero must be understood as a stochastic variable which behaves deterministically as a zero vector field. We can write  $D^{\mu\nu}{}_{\alpha\beta}(p) = D_p^{\mu\nu}{}_{\alpha\beta}(p) + D_h^{\mu\nu}{}_{\alpha\beta}(p)$ , where  $D_p^{\mu\nu}{}_{\alpha\beta}(p)$  is a particular solution to Eq. (3.55) and  $D_h^{\mu\nu}{}_{\alpha\beta}(p)$  is the most general solution to the homogeneous equation. Consequently, see Eq. (3.54), we can write  $\tilde{G}_f^{(1)\mu\nu}(p) = \tilde{G}_p^{(1)\mu\nu}(p) + \tilde{G}_h^{(1)\mu\nu}(p)$ . To find the particular solution, we try an ansatz of the form

$$D_p^{\mu\nu}{}_{\alpha\beta}(p) = d_1(p) \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu} + d_2(p) p^2 P^{\mu\nu} \eta_{\alpha\beta}. \quad (3.56)$$

Substituting this ansatz into Eqs. (3.55), it is easy to see that it solves these equations if we take

$$d_1(p) = \left[ \frac{1}{F_1(p)} \right]_r, \quad d_2(p) = - \left[ \frac{F_2(p)}{F_1(p)F_3(p)} \right]_r, \quad (3.57)$$

with

$$F_3(p) \equiv F_1(p) + 3p^2 F_2(p) = 1 - 48\pi G p^2 \left[ \tilde{H}_B(p) - 2\bar{\beta} \right], \quad (3.58)$$

and where the notation  $[ ]_r$  means that the zeros of the denominators are regulated with appropriate prescriptions in such a way that  $d_1(p)$  and  $d_2(p)$  are well defined Lorentz invariant scalar distributions. This yields a particular solution to the Einstein-Langevin equations:

$$\tilde{G}_p^{(1)\mu\nu}(p) = 16\pi G D_p^{\mu\nu}{}_{\alpha\beta}(p) \tilde{\xi}^{\alpha\beta}(p), \quad (3.59)$$

which, since the stochastic source is conserved, satisfies the conservation condition. Note that, in the case of a massless scalar field,  $m=0$ , the above solution has a functional form analogous to that of the solutions of linearized semiclassical gravity found in the Appendix of Ref. [87]. Notice also that, for a massless conformally coupled field,  $m=0$  and  $\Delta\xi=0$ , the second term in the right hand side of Eq. (3.56) will not contribute in the correlation functions (3.53), since in this case the stochastic source is traceless.

Next, we can work out the general form for  $D_h^{\mu\nu}{}_{\alpha\beta}(p)$ , which is a linear combination of terms consisting of a Lorentz invariant scalar distribution times one of the products  $\delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu}$ ,  $p^2 P^{\mu\nu} \eta_{\alpha\beta}$ ,  $\eta^{\mu\nu} \eta_{\alpha\beta}$ ,  $\eta^{\mu\nu} p^2 P_{\alpha\beta}$ ,  $\delta_{(\alpha}^{\mu} p^2 P_{\beta)}^{\nu}$  and  $p^2 P^{\mu\nu} p^2 P_{\alpha\beta}$ . However, taking into account that the stochastic source is conserved, we can omit some terms in  $D_h^{\mu\nu}{}_{\alpha\beta}(p)$  and simply write

$$\tilde{G}_h^{(1)\mu\nu}(p) = 16\pi G D_h^{\mu\nu}{}_{\alpha\beta}(p) \tilde{\xi}^{\alpha\beta}(p), \quad (3.60)$$

where

$$D_h^{\mu\nu}{}_{\alpha\beta}(p) = h_1(p) \delta_{(\alpha}^{\mu} \delta_{\beta)}^{\nu} + h_2(p) p^2 P^{\mu\nu} \eta_{\alpha\beta} + h_3(p) \eta^{\mu\nu} \eta_{\alpha\beta}, \quad (3.61)$$

with  $h_1(p)$ ,  $h_2(p)$  and  $h_3(p)$  being Lorentz invariant scalar distributions. From the fact that  $D_h^{\mu\nu}{}_{\alpha\beta}(p)$  must satisfy the homogeneous equation corresponding to Eq. (3.55), we find that  $h_1(p)$  and  $h_3(p)$  have support on the set of points  $\{p^\mu\}$  for which  $F_1(p) = 0$ , and that  $h_2(p)$  has support on the set of points  $\{p^\mu\}$  for which  $F_1(p) = 0$  or  $F_3(p) = 0$ . Moreover, the conservation condition for  $\tilde{G}_h^{(1)\mu\nu}(p)$  implies that the term with  $h_3(p)$  is only allowed in the case of a massless conformally coupled field,  $m=0$  and  $\Delta\xi=0$ . From (3.50), we get

$$\langle \tilde{G}_h^{(1)\mu\nu}(p) \tilde{\xi}^{\alpha\beta}(p') \rangle_s = (2\pi)^4 16\pi G \delta^4(p + p') D_h^{\mu\nu}{}_{\rho\sigma}(p) \tilde{N}^{\rho\sigma\alpha\beta}(p). \quad (3.62)$$

Note, from expressions (3.51) and (3.52), that the support of  $\tilde{N}^{\mu\nu\alpha\beta}(p)$  is on the set of points  $\{p^\mu\}$  for which  $-p^2 \geq 0$  when  $m=0$ , and for which  $-p^2 - 4m^2 > 0$  when  $m \neq 0$ . At such points, using expressions (3.49), (3.58), (3.39) and (3.31), it is easy to see that  $F_1(p)$  and  $F_3(p)$  are always different from zero, except for some particular values of  $\Delta\xi$  and  $\bar{\beta}$ : (a) when  $m=0$ ,  $\Delta\xi=0$  and  $\bar{\beta}>0$ ; and (b) when  $m \neq 0$ ,  $0 < \Delta\xi < (1/12)$  and  $\bar{\beta} = (\Delta\xi/32\pi^2) [\pi/(Gm^2) + 1/36]$ .

In the case (a),  $F_3(p) = 0$  for the set of points  $\{p^\mu\}$  satisfying  $-p^2 = 1/(96\pi G \bar{\beta})$ ; in the case (b),  $F_3(p) = 0$  for  $\{p^\mu\}$  such that  $-p^2 = m^2/(3\Delta\xi)$ . Hence, except for the above cases (a) and (b), the intersection of the supports of  $\tilde{N}^{\mu\nu\alpha\beta}(p)$  and  $D_h^{\rho\sigma}{}_{\lambda\gamma}(p)$  is an empty set and, thus, the correlation function (3.62) is zero. In the cases (a) and (b), we can have a contribution to (3.62) coming from the term with  $h_2(p)$  in (3.61) of the form



$D_h^{\mu\nu}{}_{\rho\sigma}(p) \tilde{N}^{\rho\sigma\alpha\beta}(p) = H_3(p; \{C\}) p^2 P^{\mu\nu} \tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$ , where  $H_3(p; \{C\})$  is the most general Lorentz invariant distribution satisfying  $F_3(p) H_3(p; \{C\}) = 0$ , which depends on a set of arbitrary parameters represented as  $\{C\}$ . However, from (3.51), we see that  $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$  is proportional to  $\theta(-p^2 - 4m^2)(1 + 4m^2/p^2)^{(1/2)}(3\Delta\xi + m^2/p^2)^2$ . Thus, in the case (a), we have  $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p) = 0$  and, in the case (b), the intersection of the supports of  $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$  and of  $H_3(p; \{C\})$  is an empty set. Therefore, from the above analysis, we conclude that  $\tilde{G}_h^{(1)\mu\nu}(p)$  gives no contribution to the correlation functions (3.53), since  $\langle \tilde{G}_h^{(1)\mu\nu}(p) \tilde{\xi}^{\alpha\beta}(p') \rangle_s = 0$ , and we have simply  $\mathcal{G}^{\mu\nu\alpha\beta}(x, x') = \langle G_p^{(1)\mu\nu}(x) G_p^{(1)\alpha\beta}(x') \rangle_s$ , where  $G_p^{(1)\mu\nu}(x)$  is the inverse Fourier transform of (3.59).

Therefore the correlation functions (3.53) can then be computed from

$$\langle \tilde{G}_p^{(1)\mu\nu}(p) \tilde{G}_p^{(1)\alpha\beta}(p') \rangle_s = 64 (2\pi)^6 G^2 \delta^4(p + p') D_p^{\mu\nu}{}_{\rho\sigma}(p) D_p^{\alpha\beta}{}_{\lambda\gamma}(-p) \tilde{N}^{\rho\sigma\lambda\gamma}(p). \quad (3.63)$$

It is easy to see from the above analysis that the prescriptions  $[\ ]_r$  in the factors  $D_p$  are irrelevant in the last expression and, thus, they can be suppressed. Taking into account that  $F_l(-p) = F_l^*(p)$ , with  $l = 1, 2, 3$ , we get from Eqs. (3.56) and (3.57)

$$\begin{aligned} \langle \tilde{G}_p^{(1)\mu\nu}(p) \tilde{G}_p^{(1)\alpha\beta}(p') \rangle_s &= 64 (2\pi)^6 G^2 \frac{\delta^4(p + p')}{|F_1(p)|^2} \left[ \tilde{N}^{\mu\nu\alpha\beta}(p) - \frac{F_2(p)}{F_3(p)} p^2 P^{\mu\nu} \tilde{N}^{\alpha\beta\rho}{}_{\rho}(p) \right. \\ &\quad \left. - \frac{F_2^*(p)}{F_3^*(p)} p^2 P^{\alpha\beta} \tilde{N}^{\mu\nu\rho}{}_{\rho}(p) + \frac{|F_2(p)|^2}{|F_3(p)|^2} p^2 P^{\mu\nu} p^2 P^{\alpha\beta} \tilde{N}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}(p) \right]. \end{aligned} \quad (3.64)$$

This last expression is well defined as a bi-distribution and can be easily evaluated using Eq. (3.51). The final explicit result for the Fourier transformed correlation function for the Einstein tensor is thus

$$\begin{aligned} \langle \tilde{G}_p^{(1)\mu\nu}(p) \tilde{G}_p^{(1)\alpha\beta}(p') \rangle_s &= \frac{2}{45} (2\pi)^5 G^2 \frac{\delta^4(p + p')}{|F_1(p)|^2} \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \\ &\quad \times \left[ \frac{1}{4} \left( 1 + 4 \frac{m^2}{p^2} \right)^2 (p^2)^2 (3P^{\mu(\alpha} P^{\beta)\nu} - P^{\mu\nu} P^{\alpha\beta}) \right. \\ &\quad \left. + 10 \left( 3\Delta\xi + \frac{m^2}{p^2} \right)^2 (p^2)^2 P^{\mu\nu} P^{\alpha\beta} \left| 1 - 3p^2 \frac{F_2(p)}{F_3(p)} \right|^2 \right]. \end{aligned} \quad (3.65)$$

To obtain the correlation functions in coordinate space, Eq. (3.53), we have to take the inverse Fourier transform of the above result, the final result is:

$$\mathcal{G}^{\mu\nu\alpha\beta}(x, x') = \frac{\pi}{45} G^2 \mathcal{F}_x^{\mu\nu\alpha\beta} \mathcal{G}_A(x - x') + \frac{8\pi}{9} G^2 \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} \mathcal{G}_B(x - x'), \quad (3.66)$$

with

$$\begin{aligned} \tilde{\mathcal{G}}_A(p) &\equiv \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left( 1 + 4 \frac{m^2}{p^2} \right)^2 \frac{1}{|F_1(p)|^2}, \\ \tilde{\mathcal{G}}_B(p) &\equiv \theta(-p^2 - 4m^2) \sqrt{1 + 4 \frac{m^2}{p^2}} \left( 3\Delta\xi + \frac{m^2}{p^2} \right)^2 \frac{1}{|F_1(p)|^2} \left| 1 - 3p^2 \frac{F_2(p)}{F_3(p)} \right|^2, \end{aligned} \quad (3.67)$$

and  $\mathcal{F}_x^{\mu\nu\alpha\beta} \equiv 3\mathcal{F}_x^{\mu(\alpha} \mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta}$ , and where  $F_l(p)$ ,  $l = 1, 2, 3$ , are given in (3.49) and (3.58). Notice that, for a massless field ( $m=0$ ), we have

$$\begin{aligned} F_1(p) &= 1 + 16\pi G p^2 \tilde{H}(p; \bar{\mu}^2), \\ F_2(p) &= -\frac{16}{3} \pi G \left[ (1 + 180\Delta\xi^2) \tilde{H}(p; \bar{\mu}^2) - 6\Upsilon \right], \\ F_3(p) &= 1 - 48\pi G p^2 \left[ 60\Delta\xi^2 \tilde{H}(p; \bar{\mu}^2) - 2\Upsilon \right], \end{aligned} \quad (3.68)$$

with  $\bar{\mu} \equiv \mu \exp(1920\pi^2 \bar{\alpha})$  and  $\Upsilon \equiv \bar{\beta} - 60\Delta\xi^2 \bar{\alpha}$ , and where  $\tilde{H}(p; \mu^2)$  is the Fourier transform of  $H(x; \mu^2)$  given in (3.39).

## 2. Correlation functions for the metric perturbations

Starting from the solutions found for the linearized Einstein tensor, which are characterized by the two-point correlation functions (3.66) [or, in terms of Fourier transforms, (3.65)], we can now solve the equations for the metric perturbations. Working in the harmonic gauge,  $\partial_\nu \bar{h}^{\mu\nu} = 0$  (this zero must be understood in a statistical sense) where  $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - (1/2)\eta_{\mu\nu}h^\alpha_\alpha$ , the equations for the metric perturbations in terms of the Einstein tensor are

$$\square \bar{h}^{\mu\nu}(x) = -2G^{(1)\mu\nu}(x), \quad (3.69)$$

or, in terms of Fourier transforms,  $p^2 \tilde{\bar{h}}^{\mu\nu}(p) = 2\tilde{G}^{(1)\mu\nu}(p)$ . Similarly to the analysis of the equation for the Einstein tensor, we can write  $\bar{h}^{\mu\nu} = \langle \bar{h}^{\mu\nu} \rangle_s + \tilde{\bar{h}}^{\mu\nu}$ , where  $\tilde{\bar{h}}^{\mu\nu}$  is a solution to these equations with zero average, and the two-point correlation functions are defined by

$$\mathcal{H}^{\mu\nu\alpha\beta}(x, x') \equiv \langle \bar{h}^{\mu\nu}(x) \bar{h}^{\alpha\beta}(x') \rangle_s - \langle \bar{h}^{\mu\nu}(x) \rangle_s \langle \bar{h}^{\alpha\beta}(x') \rangle_s = \langle \tilde{\bar{h}}^{\mu\nu}(x) \tilde{\bar{h}}^{\alpha\beta}(x') \rangle_s. \quad (3.70)$$

We can now seek solutions of the Fourier transform of Eq. (3.69) of the form  $\tilde{\bar{h}}^{\mu\nu}_f(p) = 2D(p)\tilde{G}^{(1)\mu\nu}_f(p)$ , where  $D(p)$  is a Lorentz invariant scalar distribution in Minkowski spacetime, which is the most general solution of  $p^2 D(p) = 1$ . Note that, since the linearized Einstein tensor is conserved, solutions of this form automatically satisfy the harmonic gauge condition. As in the previous subsection, we can write  $D(p) = [1/p^2]_r + D_h(p)$ , where  $D_h(p)$  is the most general solution to the associated homogeneous equation and, correspondingly, we have  $\tilde{\bar{h}}^{\mu\nu}_f(p) = \tilde{\bar{h}}^{\mu\nu}_p(p) + \tilde{\bar{h}}^{\mu\nu}_h(p)$ . However, since  $D_h(p)$  has support on the set of points for which  $p^2 = 0$ , it is easy to see from Eq. (3.65) (from the factor  $\theta(-p^2 - 4m^2)$ ) that  $\langle \tilde{\bar{h}}^{\mu\nu}_h(p) \tilde{\bar{h}}^{\alpha\beta}_h(p') \rangle_s = 0$  and, thus, the two-point correlation functions (3.70) can be computed from  $\langle \tilde{\bar{h}}^{\mu\nu}_f(p) \tilde{\bar{h}}^{\alpha\beta}_f(p') \rangle_s = \langle \tilde{\bar{h}}^{\mu\nu}_p(p) \tilde{\bar{h}}^{\alpha\beta}_p(p') \rangle_s$ . From Eq. (3.65) and due to the factor  $\theta(-p^2 - 4m^2)$ , it is also easy to see that the prescription  $[\ ]_r$  is irrelevant in this correlation function and we obtain

$$\langle \tilde{\bar{h}}^{\mu\nu}_p(p) \tilde{\bar{h}}^{\alpha\beta}_p(p') \rangle_s = \frac{4}{(p^2)^2} \langle \tilde{G}^{(1)\mu\nu}_p(p) \tilde{G}^{(1)\alpha\beta}_p(p') \rangle_s, \quad (3.71)$$

where  $\langle \tilde{G}^{(1)\mu\nu}_p(p) \tilde{G}^{(1)\alpha\beta}_p(p') \rangle_s$  is given by Eq. (3.65). The right hand side of this equation is a well defined bi-distribution, at least for  $m \neq 0$  (the  $\theta$  function provides the suitable cutoff). In the massless field case, since the noise kernel is obtained as the limit  $m \rightarrow 0$  of the noise kernel for a massive field, it seems that the natural prescription to avoid the divergencies on the lightcone  $p^2 = 0$  is a Hadamard finite part, see Refs. [90] for its definition. Taking this prescription, we also get a well defined bi-distribution for the massless limit of the last expression.

The final result for the two-point correlation function for the field  $\bar{h}^{\mu\nu}$  is:

$$\mathcal{H}^{\mu\nu\alpha\beta}(x, x') = \frac{4\pi}{45} G^2 \mathcal{F}_x^{\mu\nu\alpha\beta} \mathcal{H}_A(x - x') + \frac{32\pi}{9} G^2 \mathcal{F}_x^{\mu\nu} \mathcal{F}_x^{\alpha\beta} \mathcal{H}_B(x - x'), \quad (3.72)$$

where  $\tilde{\mathcal{H}}_A(p) \equiv [1/(p^2)^2] \tilde{\mathcal{G}}_A(p)$  and  $\tilde{\mathcal{H}}_B(p) \equiv [1/(p^2)^2] \tilde{\mathcal{G}}_B(p)$ , with  $\tilde{\mathcal{G}}_A(p)$  and  $\tilde{\mathcal{G}}_B(p)$  given by (3.67). The two-point correlation functions for the metric perturbations can be easily obtained using  $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu} - (1/2)\eta_{\mu\nu}\bar{h}^\alpha_\alpha$ .

## 3. Conformally coupled field

For a conformally coupled field, *i.e.*, when  $m = 0$  and  $\Delta\xi = 0$ , the previous correlation functions are greatly simplified and can be approximated explicitly in terms of analytic functions. The detailed results are given in Ref. [6], here we outline the main features.

When  $m = 0$  and  $\Delta\xi = 0$  we have that  $\mathcal{G}_B(x) = 0$  and  $\tilde{\mathcal{G}}_A(p) = \theta(-p^2) |F_1(p)|^{-2}$ . Thus the two-point correlations functions for the Einstein tensor is

$$\mathcal{G}^{\mu\nu\alpha\beta}(x, x') = \frac{\pi}{45} G^2 \mathcal{F}_x^{\mu\nu\alpha\beta} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-x')} \theta(-p^2)}{|1 + 16\pi G p^2 \tilde{H}(p; \bar{\mu}^2)|^2}, \quad (3.73)$$

where  $\tilde{H}(p, \mu^2) = (1920\pi^2)^{-1} \ln[-((p^0 + i\epsilon)^2 + p^i p_i)/\mu^2]$ , see Eq. (3.39).

To estimate this integral let us consider spacelike separated points  $(x - x')^\mu = (0, \mathbf{x} - \mathbf{x}')$ , and define  $\mathbf{y} = \mathbf{x} - \mathbf{x}'$ . We may now formally change the momentum variable  $p^\mu$  by the dimensionless vector  $s^\mu$ :  $p^\mu = s^\mu/|\mathbf{y}|$ , then the previous integral denominator is  $|1 + 16\pi(L_P/|\mathbf{y}|)^2 s^2 \tilde{H}(s)|^2$ , where we have introduced the Planck length  $L_P = \sqrt{G}$ . It is clear

that we can consider two regimes: (a) when  $L_P \ll |\mathbf{y}|$ , and (b) when  $|\mathbf{y}| \sim L_P$ . In case (a) the correlation function, for the 0000 component, say, will be of the order

$$\mathcal{G}^{0000}(\mathbf{y}) \sim \frac{L_P^4}{|\mathbf{y}|^8}.$$

In case (b) when the denominator has zeros a detailed calculation carried out in Ref. [6] shows that:

$$\mathcal{G}^{0000}(\mathbf{y}) \sim e^{-|\mathbf{y}|/L_P} \left( \frac{L_P}{|\mathbf{y}|^5} + \dots + \frac{1}{L_P^2 |\mathbf{y}|^2} \right)$$

which indicates an exponential decay at distances around the Planck scale. Thus short scale fluctuations are strongly suppressed.

For the two-point metric correlation the results are similar. In this case we have

$$\mathcal{H}^{\mu\nu\alpha\beta}(x, x') = \frac{4\pi}{45} G^2 \mathcal{F}_x^{\mu\nu\alpha\beta} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-x')} \theta(-p^2)}{(p^2)^2 |1 + 16\pi G p^2 \tilde{H}(p; \bar{\mu}^2)|^2}. \quad (3.74)$$

The integrand has the same behavior of the correlation function of Eq. (3.73) thus matter fields tends to suppress the short scale metric perturbations. In this case we find, as for the correlation of the Einstein tensor, that for case (a) above we have,

$$\mathcal{H}^{0000}(\mathbf{y}) \sim \frac{L_P^4}{|\mathbf{y}|^4},$$

and for case (b) we have

$$\mathcal{H}^{0000}(\mathbf{y}) \sim e^{-|\mathbf{y}|/L_P} \left( \frac{L_P}{|\mathbf{y}|} + \dots \right).$$

It is interesting to write expression (3.74) in an alternative way. If we introduce the dimensionless tensor  $P^{\mu\nu\alpha\beta} \equiv 3P^{\mu(\alpha} P^{\beta)\nu} - P^{\mu\nu} P^{\alpha\beta}$ , where  $P^{\mu\nu}$  is the projector defined in Eq. (3.24), to account for the effect of the operator  $\mathcal{F}_x^{\mu\nu\alpha\beta}$ , we can write

$$\mathcal{H}^{\mu\nu\alpha\beta}(x, x') = \frac{4\pi}{45} G^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-x')} P^{\mu\nu\alpha\beta} \theta(-p^2)}{|1 + 16\pi G p^2 \tilde{H}(p; \bar{\mu}^2)|^2}. \quad (3.75)$$

This expression allows a direct comparison with the graviton propagator for linearized quantum gravity in the  $1/N$  approximation found by Tomboulis [100]. One can see that the imaginary part of the graviton propagator leads, in fact, to Eq. (3.75).

#### 4. Discussion

The main results of this section are the correlation functions (3.66) and (3.72). In the case of a conformal field, the correlation functions of the linearized Einstein tensor have been explicitly estimated. From the exponential factors  $e^{-|\mathbf{y}|/L_P}$  in these results for scales near the Planck length, we see that the correlation functions of the linearized Einstein tensor have the Planck length as the correlation length. A similar behavior is found for the correlation functions of the metric perturbations. Since these fluctuations are induced by the matter fluctuations we infer that the effect of the matter fields is to suppress the fluctuations of the metric at very small scales. On the other hand, at scales much larger than the Planck length the induced metric fluctuations are small compared with the free graviton propagator which goes like  $L_P^2/|\mathbf{y}|^2$ , since the action for the free graviton goes like  $S_h \sim \int d^4 x L_P^{-2} h \square h$ .

It is interesting to note, however, that these results for correlation functions are non-analytic in their characteristic correlation lengths. This kind of non-analytic behavior is actually quite typical of the solutions of Langevin-type equations with dissipative terms. An example in the context of a reduced version of the semiclassical Einstein-Langevin equation is given in Ref. [86].

For background solutions of semiclassical gravity with other scales present apart from the Planck scales (for instance, for matter fields in a thermal state), stress-energy fluctuations may be important at larger scales. For such backgrounds, stochastic semiclassical gravity might predict correlation functions with characteristic correlation lengths

larger than the Planck scales. It seems quite plausible, nevertheless, that these correlation functions would remain non-analytic in their characteristic correlation lengths. This would imply that these correlation functions could not be obtained from a calculation involving a perturbative expansion in the characteristic correlation lengths. In particular, if these correlation lengths are proportional to the Planck constant  $\hbar$ , the gravitational correlation functions could not be obtained from an expansion in  $\hbar$ . Hence, stochastic semiclassical gravity might predict a behavior for gravitational correlation functions different from that of the analogous functions in perturbative quantum gravity [80]. This is not necessarily inconsistent with having neglected action terms of higher order in  $\hbar$  when considering semiclassical gravity as an effective theory [87].

#### IV. FLUCTUATIONS OF ENERGY DENSITY AND VALIDITY OF SEMICLASSICAL GRAVITY

We now turn our attention to some basic issues involving vacuum energy density fluctuations invoking only the simplest spacetimes, Minkowski and Casimir. Recent years saw the beginning of serious studies of the fluctuations of the energy momentum tensor (EMT)  $\hat{T}_{\mu\nu}$  of quantum fields in spacetimes with boundaries [20–22] (such as Casimir effect [101]) [102,61], nontrivial topology (such as imaginary time thermal field theory) or nonzero curvature (such as the Einstein universe) [62]. A natural measure of the strength of fluctuations is  $\chi$  [7], the ratio of the variance  $\Delta\rho^2$  of fluctuations in the energy density (expectation value of the  $\hat{\rho}^2$  operator minus the square of the mean  $\hat{\rho}$  taken with respect to some quantum state) to its mean-squared (square of the expectation value of  $\hat{\rho}$ ):

$$\chi \equiv \frac{\langle \hat{\rho}^2 \rangle - \langle \hat{\rho} \rangle^2}{\langle \hat{\rho} \rangle^2} \equiv \frac{\Delta\rho^2}{\bar{\rho}^2} \quad (4.1)$$

Alternatively, we can use the quantity introduced by Kuo and Ford [61]

$$\Delta \equiv \frac{\langle \hat{\rho}^2 \rangle - \langle \hat{\rho} \rangle^2}{\langle \hat{\rho}^2 \rangle} = \frac{\chi}{\chi + 1} \quad (4.2)$$

Assuming a positive definite variance  $\Delta\rho^2 \geq 0$ , then  $0 \leq \chi \leq \infty$  and  $0 \leq \Delta \leq 1$  always, with  $\Delta \ll 1$  falling in the classical domain. Kuo and Ford (KF) displayed a number of quantum states (vacuum plus 2 particle state, squeezed vacuum and Casimir vacuum) with respect to which the expectation value of the energy momentum tensor (00 component) gives rise to negative local energy density. For these states  $\Delta$  is of order unity. From this result they drew the implications, amongst other interesting inferences, that semiclassical gravity (SCG) [24] could become invalid under these conditions. The validity of semiclassical gravity in the face of fluctuations of quantum fields as source is an important issue which has caught the attention of many authors. Amongst others Phillips and Hu (PH) [8] hold a different viewpoint on this issue from KF. This section is a summary of their investigations on this issue.

To begin with it may not be so surprising that states which are more quantum (e.g., squeezed states) in nature than classical (e.g., coherent states) [103] may lead to large fluctuations in energy density comparable to the mean<sup>1</sup>. Such a condition exists peacefully with the underlying spacetime at least at the low energy of today's universe. PH calculated the variance of fluctuations to mean-squared ratio of a quantum field for the simplest case of Minkowski spacetime i.e., for ordinary quantum field theory to be  $\Delta = 2/5$ . This is a simple counter-example to the claim of KF, since  $\Delta = O(1)$  holds also for Minkowski space, where SCG is known to be valid at large scales. PH do not see sufficient ground to question the validity of SCG at energy below the Planck energy when the spacetime is depictable by a manifold structure, approximated locally by the Minkowski space. To them the fluctuations to mean being of the order unity arises from the quantum nature of the vacuum state and says little about the compatibility of the field source with the spacetime the quantum field lives in<sup>2</sup>.

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<sup>1</sup>This can be seen even in the ratio of expectation values of moments of the displacement operators in simple quantum harmonic oscillators

<sup>2</sup>One should draw a distinction between quantum fields in curved spacetime QFCST and semiclassical gravity: the former is a test field situation with quantum fields propagating in a fixed background space while in the latter both the field and the spacetime are determined self-consistently by solving the semiclassical Einstein equation. The cases studied in Kuo and Ford [61] as well as many others [62] are of a test-field nature, where backreaction is not considered. So KF's criterion pertains more to QFTCST than to SCG, where in the former the central issue is compatibility, which is a weaker condition than consistency in the latter.

PH pointed out that one should refer to a scale (of interaction or for probing accuracy) when measuring the validity of SCG. The conventional belief is that when reaching the Planck scale from below, QFTCST will break down because, amongst other things happening, graviton production at that energy will become significant so as to render the classical background spacetime unstable, and the mean value of quantum field taken as a source for the Einstein equation becomes inadequate. To address this issue as well as the issue of the spatial extent where negative energy density can exist, PH view it necessary to introduce a scale in the spacetime regions where quantum fields are defined to monitor how the mean value and the fluctuations of the energy momentum tensor change.

In conventional field theories the stress tensor built from the product of a pair of field operators evaluated at a single point in the spacetime manifold is, strictly speaking, ill-defined. Point separation is a well-established method which suits the present concern very well, and we will discuss this method in section V. For here we will use a simpler method to introduce a scale in the quantum field theory, i.e., by introducing a (spatial) smearing function  $f(\mathbf{x})$  to define smeared field operators  $\hat{\phi}_t(f_{\mathbf{x}})$ . Using a Gaussian smearing function (with variance  $\sigma^2$ ) PH derive expressions for the EM bi-tensor operator, its mean and its fluctuations as functions of  $\sigma$ , for a massless scalar field in both the Minkowski and the Casimir spacetimes. The interesting result PH find is that while both the vacuum expectation value and the fluctuations of energy density grow as  $\sigma \rightarrow 0$ , the ratio of the variance of the fluctuations to its mean-squared remains a constant  $\chi_d$  ( $d$  is the spatial dimension of spacetime) which is independent of  $\sigma$ . The measure  $\Delta_d (= \chi_d/(\chi_d + 1))$  depends on the dimension of space and is of the order unity. It varies only slightly for spacetimes with boundary or nontrivial topology. For example  $\Delta$  for Minkowski is  $2/5$ , while for Casimir is  $6/7$  (cf, from [62]). Add to this our prior result for the Einstein Universe,  $\Delta = 111/112$ , independent of curvature, and that for hot flat space [104], we see that invariably the fluctuations to mean ratio is of the order unity.

These results allow us to address three interrelated issues in quantum field theory in curved spacetime in the light of fluctuations of quantum stress energy : 1) Fluctuation to mean ratio of vacuum energy density and the validity of semiclassical gravity. 2) The spatial extent where negative energy density can exist and its implications for quantum effects of worm holes, baby universes and time travel. 3) Dependence of fluctuations on intrinsic (defined by smearing or point-separation) and the extrinsic scale (such as the Casimir or finite temperature periodicity). 4) The circumstances when and how divergences appear and the meaning of regularization in point-defined field theories versus theories defined at separated points and/or smeared fields. This includes also the issue of the cross term.

We begin by defining the smeared field operators and their products and construct from them the smeared energy density and its fluctuations. We then calculate the ratio of the fluctuations to the mean for a flat space (Minkowski geometry) followed by a Casimir geometry of one periodic spatial dimension. Finally we discuss the meaning of our finding in relation to the issues raised above.

### A. Smeared Field Operators

Since the field operator in conventional point-defined quantum field theory is an operator-valued distribution, products of field operators at a point become problematic. This parallels the problem with defining the square of a delta function  $\delta^2(x)$ . Distributions are defined via their integral against a test function: they live in the space dual to the test function space. By going from the field operator  $\hat{\phi}(x)$  to its integral against a test function,  $\hat{\phi}(f) = \int \hat{\phi} f$ , we can now readily consider products.

When we take the test functions to be spatial Gaussians, we are smearing the field operator over a finite spatial region. Physically we see smearing as representing the necessarily finite extent of an observer's probe, or the intrinsic limit of resolution in carrying out a measurement at a low energy (compared to Planck scale). In contrast to the ordinary point-defined quantum field theory, where ultraviolet divergences occur in the energy momentum tensor, smeared fields give no ultraviolet divergence. This is because smearing is equivalent to a regularization scheme which imparts an exponential suppression to the high momentum modes and restricts the contribution of the high frequency modes in the mode sum.

With this in mind, we start by defining the spatially smeared field operator

$$\hat{\phi}_t(f_{\mathbf{x}}) = \int \hat{\phi}(t, \mathbf{x}') f_{\mathbf{x}}(\mathbf{x}') d\mathbf{x}' \quad (4.3)$$

where  $f_{\mathbf{x}}(\mathbf{x}')$  is a suitably smooth function. With this, the two point operator becomes

$$\left(\hat{\phi}_t(f_{\mathbf{x}})\right)^2 = \int \int \hat{\phi}(t, \mathbf{x}') \hat{\phi}(t, \mathbf{x}'') f_{\mathbf{x}}(\mathbf{x}') f_{\mathbf{x}}(\mathbf{x}'') d\mathbf{x} d\mathbf{x}'' \quad (4.4)$$

which is now finite. In terms of the vacuum  $|0\rangle$  ( $\hat{a}_{\mathbf{k}}|0\rangle = 0$ , for all  $\mathbf{k}$ ) we have the usual mode expansion

$$\hat{\phi}(t_1, \mathbf{x}_1) = \int d\mu(\mathbf{k}_1) \left( \hat{a}_{\mathbf{k}_1} u_{\mathbf{k}_1}(t_1, \mathbf{x}_1) + \hat{a}_{\mathbf{k}_1}^\dagger u_{\mathbf{k}_1}^*(t_1, \mathbf{x}_1) \right) \quad (4.5)$$

with

$$u_{\mathbf{k}_1}(t_1, \mathbf{x}_1) = N_{k_1} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 - t_1 \omega_1)}, \quad \omega_1 = |\mathbf{k}_1|, \quad (4.6)$$

where the integration measure  $\int d\mu(\mathbf{k}_1)$  and the normalization constants  $N_{\mathbf{k}_1}$  are given for a Minkowski and Casimir spaces by (4.15) and (4.21) respectively.

Consider a Gaussian smearing function

$$f_{\mathbf{x}_0}(\mathbf{x}) = \left( \frac{1}{4\pi\sigma^2} \right)^{\frac{d}{2}} e^{-\left( \frac{\mathbf{x}_0 - \mathbf{x}}{2\sigma} \right)^2} \quad (4.7)$$

with the properties  $\int f_{\mathbf{x}_0}(\mathbf{x}') d\mathbf{x}' = 1$ ,  $\int \mathbf{x}' f_{\mathbf{x}_0}(\mathbf{x}') d\mathbf{x}' = \mathbf{x}_0$  and  $\int |\mathbf{x}'|^2 f_{\mathbf{x}_0}(\mathbf{x}') d\mathbf{x}' = 2d\sigma^2 + |\mathbf{x}_0|^2$ . Using

$$\begin{aligned} \int u_{\mathbf{k}_1}(t, \mathbf{x}) f_{\mathbf{x}_1}(\mathbf{x}) d\mathbf{x} &= N_{k_1} e^{-it\omega_1} \prod_{i=1}^d \left( \frac{1}{2\sqrt{\pi}\sigma} \int e^{+ik_{1i}x_i - \left( \frac{x_{1i} - x_i}{2\sigma} \right)^2} dx_i \right) \\ &= N_{k_1} e^{-it\omega_1 + i\mathbf{k}_1 \cdot \mathbf{x}_1 - \sigma^2 k_1^2} \end{aligned} \quad (4.8)$$

we get the smeared field operator

$$\hat{\phi}_{t_1}(f_{\mathbf{x}_1}) = \int d\mu(\mathbf{k}_1) N_{k_1} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1 - \sigma^2 k_1^2 - it_1 \omega_1} \left( e^{2i\mathbf{k}_1 \cdot \mathbf{x}_1} \hat{a}_{\mathbf{k}_1} + e^{2it_1 \omega_1} \hat{a}_{\mathbf{k}_1}^\dagger \right) \quad (4.9)$$

and their derivatives

$$\begin{aligned} \left( \partial_{t_1} \hat{\phi}_{t_1} \right) (f_{\mathbf{x}_1}) &= \int \left( \partial_{t_1} \hat{\phi}(t_1, \mathbf{x}') \right) f_{\mathbf{x}_1}(\mathbf{x}') d\mathbf{x}' \\ &= i \int d\mu(\mathbf{k}_1) N_{k_1} \omega_1 e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1 - \sigma^2 k_1^2 - it_1 \omega_1} \left( e^{2it_1 \omega_1} \hat{a}_{\mathbf{k}_1}^\dagger - e^{2i\mathbf{k}_1 \cdot \mathbf{x}_1} \hat{a}_{\mathbf{k}_1} \right) \end{aligned} \quad (4.10a)$$

$$\begin{aligned} \left( \vec{\nabla}_{\mathbf{x}_1} \hat{\phi}_{t_1} \right) (f_{\mathbf{x}_1}) &= \int \left( \vec{\nabla}_{\mathbf{x}'} \hat{\phi}(t_1, \mathbf{x}') \right) f_{\mathbf{x}_1}(\mathbf{x}') d\mathbf{x}' \\ &= -i \int d\mu(\mathbf{k}_1) \mathbf{k}_1 N_{k_1} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1 - \sigma^2 k_1^2 - it_1 \omega_1} \left( e^{2it_1 \omega_1} \hat{a}_{\mathbf{k}_1}^\dagger - e^{2i\mathbf{k}_1 \cdot \mathbf{x}_1} \hat{a}_{\mathbf{k}_1} \right). \end{aligned} \quad (4.10b)$$

From this we can calculate the two point function of the field which make up the energy density and their correlation function. Letting  $x \equiv (t, \mathbf{x}) = (t_2, \mathbf{x}_2) - (t_1, \mathbf{x}_1)$ , they are given by

$$\rho(t, \mathbf{x}; \sigma) = \int d\mu(\mathbf{k}) N_k^2 \omega^2 e^{-2\sigma^2 k^2} \cos(\mathbf{x} \cdot \mathbf{k} - t\omega) \quad (4.11)$$

$$\Delta\rho^2(t, \mathbf{x}; \sigma) = \frac{1}{2} \int d\mu(\mathbf{k}_1, \mathbf{k}_2) N_{k_1}^2 N_{k_2}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2 + \omega_1 \omega_2)^2 e^{-2\sigma^2 (k_1^2 + k_2^2) - i\mathbf{x} \cdot (\mathbf{k}_1 + \mathbf{k}_2) + it(\omega_1 + \omega_2)} \quad (4.12)$$

Setting  $x = 0$  we obtain the smeared vacuum energy density at one point

$$\rho(\sigma) = \int d\mu(\mathbf{k}_1) N_{k_1}^2 \omega_1^2 e^{-2\sigma^2 k_1^2} \quad (4.13)$$

### 1. Smeared-Field Energy Density and Fluctuations in Minkowski Space

We consider a Minkowski space  $R^1 \times R^d$  with  $d$ -spatial dimensions. For this space the mode density is

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-1} dk \int_{S^{d-1}} d\Omega_{d-1} \quad \text{with} \quad \int_{S^{d-1}} d\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \quad (4.14)$$

and the mode function normalization constant is

$$N_{k_1} = 1/\sqrt{2^{d+1} \pi^d \omega_1}. \quad (4.15)$$

We introduce the angle between two momenta in phase space,  $\gamma$ , via

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = k_1 k_2 \cos(\gamma) = \omega_1 \omega_2 \cos(\gamma). \quad (4.16)$$

The averages of the cosine and cosine squared of this angle over a pair of unit spheres are

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos(\gamma) = 0 \quad (4.17a)$$

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos^2(\gamma) = \frac{4 \pi^d}{d \Gamma\left(\frac{d}{2}\right)^2}. \quad (4.17b)$$

The smeared energy density (4.13) becomes

$$\begin{aligned} \rho(\sigma) &= \frac{1}{2^d \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{k_1^d}{e^{2\sigma^2 k_1^2}} dk_1 \\ &= \frac{\Gamma\left(\frac{d+1}{2}\right)}{2^{\frac{3(d+1)}{2}} \pi^{\frac{d}{2}} \sigma^{d+1} \Gamma\left(\frac{d}{2}\right)} \end{aligned} \quad (4.18)$$

For the fluctuations of the smeared energy density operator, we evaluate (4.12) for this space and find

$$\begin{aligned} \Delta\rho^2(\sigma) &= \frac{1}{2^{(2d+3)} \pi^{2d}} \int_0^\infty \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} \frac{(1 + \cos(\gamma))^2 k_1^d k_2^d}{e^{2\sigma^2 (k_1^2 + k_2^2)}} d\Omega_1 d\Omega_2 dk_1 dk_2 \\ &= \frac{(d+1) \Gamma\left(\frac{d+1}{2}\right)^2}{2^{(3d+4)} d \pi^d \sigma^{2(d+1)} \Gamma\left(\frac{d}{2}\right)^2} \end{aligned} \quad (4.19)$$

Putting these together we obtain for the Minkowski space

$$\Delta_{\text{Minkowski}}(d) = \frac{1+d}{1+3d} \quad (4.20)$$

which has the particular values:

$$\frac{1}{2} \text{ for } d=1, \quad \frac{2}{5} \text{ for } d=3, \quad \frac{3}{8} \text{ for } d=5, \quad \frac{1}{3} \text{ for } d=\infty.$$

## 2. Smeared-Field in Casimir Topology

The Casimir topology is obtained from a flat space (with  $d$  spatial dimensions, i.e.,  $R^1 \times R^d$ ) by imposing periodicity  $L$  in one of its spatial dimensions, say,  $z$ , thus endowing it with a  $R^1 \times R^{d-1} \times S^1$  topology. We decompose  $\mathbf{k}$  into a component along the periodic dimension and call the remaining components  $\mathbf{k}_\perp$ :

$$\mathbf{k} = \left( \mathbf{k}_\perp, \frac{2\pi n}{L} \right) = (\mathbf{k}_\perp, l n), l \equiv 2\pi/L \quad (4.21a)$$

$$\omega_1 = \sqrt{k_\perp^2 + l^2 n_1^2} \quad (4.21b)$$

The normalization and momentum measure are

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-2} dk \int_{S^{d-2}} d\Omega_{d-2} \sum_{n=-\infty}^\infty \quad (4.21c)$$

$$N_{k_1} = \frac{1}{\sqrt{2^d L \pi^{d-1} \omega_1}} \quad (4.21d)$$

With this, the energy density (4.13) becomes

$$\rho_L(\sigma) = \frac{l}{2^d \pi^{\frac{d+1}{2}} \Gamma(\frac{d-1}{2})} \sum_{n_1=-\infty}^{\infty} \int_0^{\infty} k_1^{d-2} (k_1^2 + l^2 n_1^2)^{\frac{1}{2}} e^{-2\sigma^2 (k_1^2 + l^2 n_1^2)} dk_1 \quad (4.22)$$

we can write this as the sum of the two smeared Green function derivatives

$$\begin{aligned} \rho_L(\sigma) &= \left\langle 0_L \left| ((\nabla_{\perp} \phi_t)(f_{\mathbf{x}}))^2 \right| 0_L \right\rangle + \left\langle 0_L \left| ((\partial_z \phi_t)(f_{\mathbf{x}}))^2 \right| 0_L \right\rangle \\ &= G_L(\sigma)_{,x_{\perp}x_{\perp}} + G_L(\sigma)_{,zz} \end{aligned} \quad (4.23)$$

where  $|0_L\rangle$  is the Casimir vacuum.

Since  $G_L(\sigma)_{,i} = G_L^{\text{div}} + G_L^{\text{fin}}_{,i}$  ( $i = x_{\perp}x_{\perp}$  or  $zz$ ) we see how to split the smeared energy density into a  $\sigma \rightarrow 0$  divergent term and the finite contribution:

$$\rho_L(\sigma) = \rho_L^{\text{div}} + \rho_L^{\text{fin}} \quad (4.24)$$

where

$$\begin{aligned} \rho_L^{\text{div}} &= G_L^{\text{div}}_{,x_{\perp}x_{\perp}} + G_L^{\text{div}}_{,zz} \\ &= \frac{\Gamma(\frac{d+1}{2})}{2^{\frac{3(d+1)}{2}} \pi^{\frac{d}{2}} \sigma^{d+1} \Gamma(\frac{d}{2})} \\ &= \rho(\sigma) \end{aligned} \quad (4.25)$$

and

$$\begin{aligned} \rho_L^{\text{fin}} &= G_L^{\text{fin}}_{,x_{\perp}x_{\perp}} + G_L^{\text{fin}}_{,zz} \\ &= -\frac{d \Gamma(-\frac{d}{2}) \Gamma(\frac{d}{2})}{(4\pi)^{(d+3)/2} l^{d+1}} \sum_{p=1}^{\infty} (-1)^p (2l)^{2p} p (2p-1)^2 \sigma^{2(p-1)} \frac{B_{2p+d-1}}{2p+d-1} \frac{(2p-3)!! \Gamma(p-\frac{1}{2})}{(2p)! \Gamma(p+\frac{d}{2})} \end{aligned} \quad (4.26)$$

With this we define the regularized energy density

$$\begin{aligned} \rho_{L,\text{reg}} &\equiv \lim_{\sigma \rightarrow 0} (\rho_L(\sigma) - \rho(\sigma)) \\ &= \frac{d \pi^{\frac{d}{2}} B_{d+1} \Gamma(-\frac{d}{2}) \Gamma(\frac{d}{2})}{2(d+1) L^{d+1} \Gamma(\frac{d}{2} + 1)} \end{aligned} \quad (4.27)$$

and get the usual results:

$$-\frac{\pi}{6L^2} \text{ for } d=1, \quad -\frac{\pi^2}{90L^4} \text{ for } d=3, \quad -\frac{2\pi^3}{945L^6} \text{ for } d=5.$$

For the  $d$ -dimensional Casimir geometry, the fluctuations are

$$\begin{aligned} \Delta \rho_L^2(\sigma) &= \frac{l^2}{2^{2d+3} \pi^{2d}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \int_0^{\infty} k_1^{d-2} dk_1 \int_0^{\infty} k_2^{d-2} dk_2 \int_{S^{d-2}} d\Omega_1 \int_{S^{d-2}} d\Omega_2 \\ &\quad \times \frac{e^{-2\sigma^2(\omega_1^2 + \omega_2^2)}}{\omega_1 \omega_2} (\cos(\gamma) k_1 k_2 + l^2 n_1 n_2 + \omega_1 \omega_2)^2 \end{aligned} \quad (4.28)$$

This expression can again be written in terms of products of the Green functions derivatives used above:

$$\Delta \rho_L^2(\sigma) = \frac{d \left( G_L(\sigma)_{,x_{\perp}x_{\perp}} \right)^2}{2(d-1)} + G_L(\sigma)_{,zz} \left( G_L(\sigma)_{,x_{\perp}x_{\perp}} + G_L(\sigma)_{,zz} \right) \quad (4.29)$$

and split into three general terms

$$\Delta \rho_L^2(\sigma) = \Delta \rho_L^{2,\text{div}} + \Delta \rho_L^{2,\text{cross}} + \Delta \rho_L^{2,\text{fin}} \quad (4.30)$$



The full expression for each of these terms are given in the Appendix of [8]. The first term contains only the divergent parts of the Green functions while the last term contains only the finite parts. This is similar to the split we used for the smeared energy density above. What is new here is the middle term  $\Delta\rho_L^{2,\text{cross}}$ . This comes about from the products of the divergent part of one Green function and the finite part of the other. That this term arises for computations of the energy density fluctuations is a generic feature. We will discuss in greater detail the meaning of this term later.

The results of Appendix A in [8] give

$$\Delta\rho_L^{2,\text{div}} = \chi_d (\rho_L^{\text{div}})^2 = \chi_d (\rho(\sigma))^2, \quad (4.31a)$$

$$\Delta\rho_L^{2,\text{cross}} = 2\chi_d \rho_L^{\text{div}} \rho_L^{\text{fin}}, \quad (4.31b)$$

$$\Delta\rho_L^{2,\text{fin}} = \frac{d(d+1)}{2} (\rho_{L,\text{reg}})^2. \quad (4.31c)$$

From this we see the divergent and cross terms can be related to the smeared energy density via

$$\Delta\rho_L^{2,\text{div}} + \Delta\rho_L^{2,\text{cross}} = \chi_d \left\{ (\rho_L^{\text{div}})^2 + 2\rho_L^{\text{div}} \rho_L^{\text{fin}} \right\} \quad (4.32)$$

where  $\chi_d$  is the function that relates the fluctuations of the energy density to the mean energy density when the boundaries are not present, i.e., Minkowski space. This leads us to interpret these terms as due to the vacuum fluctuations that are always present. With this in mind, we define the regularized fluctuations of the energy density

$$\begin{aligned} \Delta\rho_{L,\text{reg}}^2 &= \lim_{\sigma \rightarrow 0} \left( \Delta\rho_L^2(f) - \chi_d \left\{ (\rho_L^{\text{div}})^2 + 2\rho_L^{\text{div}} \rho_L^{\text{fin}} \right\} \right) \\ &= \chi_{d,L} (\rho_{L,\text{reg}})^2 \end{aligned} \quad (4.33)$$

where

$$\chi_{d,L} \equiv \frac{d(d+1)}{2}. \quad (4.34)$$

We also define a regularized version of the dimensionless measure  $\Delta$ :

$$\Delta_{L,\text{Reg}} \equiv \frac{\Delta\rho_{L,\text{Reg}}^2}{\Delta\rho_{L,\text{Reg}}^2 + (\rho_{L,\text{Reg}})^2} = \frac{d(d+1)}{2+d+d^2} \quad (4.35)$$

and note the values:

$$\frac{1}{2} \text{ for } d=1, \quad \frac{6}{7} \text{ for } d=3, \quad \frac{15}{16} \text{ for } d=5, \quad 1 \text{ for } d=\infty.$$

Following the procedures described in Appendix B of [8], Phillips and Hu have made two plots, Fig. 1 of  $\Delta(\sigma, L)$  and  $\Delta_{L,\text{Reg}}$  versus  $\sigma/L$ , (which we call  $\sigma'$  here for short); and Fig. 2 of  $\rho_{L,\text{Reg}}$  and  $\sqrt{\Delta\rho_{L,\text{Reg}}^2}$  versus  $\sigma'$ . The range of  $\sigma'$  is limited to  $\leq 0.4$  because going any further would make the meaning of a local energy density ill-defined, as the smearing of the field extends to the Casimir boundary in space. (The infrared limit also carry important physical meaning in reference to the structure of spacetime.)

Let us ponder on the meaning they convey. In Fig. 1, we first note that both curves are of the order unity. But the behavior of  $\Delta$  (recall that the energy density fluctuations thus defined include the cross term along with the finite part and the state independent divergent part) is relatively insensitive to the smearing width, whereas  $\Delta_{L,\text{Reg}}$ , which measures only the finite part of the energy density fluctuations to the mean, has more structure. In particular, it saturates its upper bound of 1 around  $\sigma' = 0.24$ . Note that if one adheres to the KF criterion [61] one would say that semiclassical gravity fails, but all that is happening here is that  $\rho_{L,\text{Reg}} = 0$  while  $\Delta\rho_{L,\text{Reg}}^2$  shows no special feature. The real difference between these two functions is the cross term, which is responsible for their markedly different structure and behavior.

In Fig. 2, the main feature to notice is that the regularized energy density crosses from negative to positive values at around  $\sigma' = 0.24$ . The negative Casimir energy density calculated in a point-wise field theory which corresponds to small ranges of  $\sigma'$  is expected, and is usually taken to signify the quantum nature of the Casimir state. As  $\sigma'$  increases we are averaging the field operator over a larger region, and thus sampling the field theory from the

ultraviolet all the way to the infrared region. At large  $\sigma'$  finite size effect begins to set in. The difference and relation of these two effects are explained in [105]: Casimir effect arises from summing up the quantum fluctuations of ALL modes (as altered by the boundary), with no insignificant short wavelength contributions, whereas finite size effect has dominant contributions from the LONGEST wavelength modes, and thus reflect the large scale behavior. As the smearing moves from a small scale to the far boundary of space, the behavior of the system is expected to shift from a Casimir-dominated to a finite size-dominated effect. This could be the underlying reason in the crossover behavior of  $\rho_{L,Reg}$ .

## B. Fluctuation to Mean Ratio and Spatial Extent of Negative Energy Density

Now that we have the results we can return to the issues raised earlier. We discuss the first two here, i.e., 1) Fluctuations of the energy density and validity of semiclassical gravity, 2) The spatial extent where negative energy density can exist. We will discuss the regularization of energy density fluctuations and the issue of the cross term in the next subsection.

### 1. Fluctuation to Mean ratio and Validity of SCG

From these results we see that i) the fluctuations of the energy density as well as its mean both increase with decreasing distance (or probing scale), while ii) the ratio of the variance of the fluctuations in EMT to its mean-squared is of the order unity. We view the first but not the second feature as linked to the question of the validity of SCG. The second feature represents something quite different, pertaining more to the quantum nature of the vacuum state than to the validity of SCG.

If we adopt the criterion of Kuo and Ford [61] that the variance of the fluctuation relative to the mean-squared (vev taken with respect to the ordinary Minkowskian vacuum) being of the order unity be an indicator of the failure of SCG, then all spacetimes studied above would indiscriminately fall into that category, and SCG fails wholesale, regardless of the scale these physical quantities are probed. This contradicts with the common expectation that SCG is valid at scales below Planck energy. We believe the criterion for the validity or failure of a theory should depend on the range or the energy probed. The findings of Phillips and Hu [8] related here seem to confirm this: Both the mean (the vev of EMT with respect to the Minkowski vacuum) AND the fluctuations of EMT increase as the scale decreases. As one probes into an increasingly finer scale or higher energy the expectation value of EMT grows in value and the induced metric fluctuations become important, leading to the failure of SCG. A generic scale for this to ‘happen is the Planck length. At such energy densities and above, particle creation from the quantum field vacuum would become copious and their backreaction on the background spacetime would become important [24]. Fluctuations in the quantum field EMT entails these quantum processes. The induced metric fluctuations [3] render the smooth manifold structure of spacetime inadequate, spacetime foams [106] including topological transitions [107] begin to appear and SCG no longer can provide an adequate description of these dominant processes. This picture first conjured by Wheeler is consistent with the common notion adopted in SCG, and we believe it is a valid one.

### 2. Extent of Negative Energy Density

It is well known that negative energy density exists in Casimir geometry, moving mirrors, black holes and worm holes. Proposals have also been conjured to use the negative energy density for the design of time machines [108]. Our results (Figures 1, 2) provide an explicit scale dependence of the regularized vacuum energy density  $\rho_{L,reg}$  and its fluctuations  $\Delta_{L,reg}$ , specifically  $\sigma/L$ , the ratio of the smearing length (field scale) to that of the Casimir length (geometry scale). For example, Fig. 2 shows that only for  $\sigma/L < 0.24$  is  $\rho_{L,reg} < 0$ . Recall  $\sigma$  gives the spatial extent the field is probed or smeared. Ordinary pointwise quantum field theory which probes the field only at a point does not carry information about the spatial extent where negative energy density sustains. These results have direct implications on wormhole physics (and time machines, if one gets really serious about these fictions [108]). If  $L$  is the scale characterizing the size (‘throat’) of the wormhole where one thinks negative energy density might prevail, and designers of ‘time machines’ wish to exploit for ‘time-travel’, the results of Phillips and Hu can provide a limit on the size of the probe (spaceship in the case of time-travel) in ratio to  $L$  where such conditions may exist. It could also provide a quantum field-theoretical bound on the probability of spontaneous creation of baby universes from quantum field energy fluctuations.

### C. Dependence of fluctuations on intrinsic and extrinsic scales

One may ask why the fluctuations of the energy density to its mean for the many cases calculated by PH and KF should be the fractional numbers as they are. Is there any simple reason behind the following features observed in Phillips and Hu's calculations?

- a)  $\Delta = O(1)$
- b) The specific numeric values of  $\Delta$  for the different cases.
- c) For the Minkowski vacuum the ratio of the variance to the mean-squared, calculated from the coincidence limit, is identical to the value of the Casimir case at the same limit for spatial point separation while identical to the value of a hot flat space result with a temporal point-separation.

Point a) has also been shown by earlier calculations [61,62], and our understanding is that this is true only for states of quantum nature, including the vacuum and certain squeezed states, but probably not true for states of a more classical nature like the coherent state. We also emphasized that this result should not be used as a criterion for the validity of semiclassical gravity.

For point b), we can trace back the calculation of the fluctuations (second moment) of the energy momentum tensor in ratio to its mean (first moment) to the integral of the term containing the inner product of two momenta  $\mathbf{k}_1 \cdot \mathbf{k}_2$  summed over all participating modes. The modes contributing to this are different for different geometries, e.g., Minkowski versus Casimir boundary –for the Einstein universe this enters as 3j symbols – and could account for the difference in the numerical values of  $\Delta$  for the different cases.

For point c), to begin with, it is well-known that the regularization by taking the coincidence limit of a spatial versus a temporal point separation will give different results. The case of temporal split involves integration of three spatial dimensions while the case of spatial split involve integration of two remaining spatial and one temporal dimension, which would give different results. The calculation of fluctuations involves the second moment of the field and is in this regard similar to what enters into the calculation of moments of inertia for rotating objects. We suspect that the difference between the temporal and the spatial results is similar, to the extent this analogy holds, to the difference in the moment of inertia of the same object but taken with respect to different axes of rotation.

It may be surprising that in a Minkowski calculation the result of Casimir geometry or thermal field should appear, as both cases involve a scale – the former in the spatial dimension and the latter in the (imaginary) temporal dimension. (Both cases have the same topology  $R^3 \times S^1$ , with the  $S^1$  in the (imaginary) time for the former and in the space for the latter.) But if we note that the results for Casimir geometry or thermal field are obtained at the coincidence (ultraviolet) limit, when the scale (infrared) of the problem does not intercede in any major way, then the main components of the calculations for these two cases would be similar to the two cases of taking the coincidence limit in the spatial and temporal directions in Minkowski space.

All of these cases are effectively devoid of scale as far as the pointwise field theory is concerned. As soon as we depart from this limit the effect of the presence of a scale shows up. The point-separated or field-smeared results for the Casimir calculation shows clearly that the boundary scale enters in a major way and the result for the fluctuations and the ratio are different from those obtained at the coincident limit. For other cases where a scale enters intrinsically in the problem such as that of a massive or non-conformally coupled field it would show a similar effect in these regards as the present cases (of Casimir and thermal field) where a periodicity condition exists (in the spatial and temporal directions respectively). We expect a similar strong disparity between point-coincident and point-separated cases. The field theory changes its nature in a fundamental and physical way when this limit is taken. This brings us to an even more fundamental issue made clear in this investigation, i.e., the appearance of divergences and the meaning of regularization in the light of a point-separated versus a point-defined quantum field theory.

### D. Regularization in the Fluctuations of EMT and the Issue of the Cross Term

An equally weighty issue brought to light in the study of Phillips and Hu is the meaning of regularization in the face of EMT fluctuations. Since the point-separated or smeared field expressions of the EMT and its fluctuations become available one can study how they change as a function of separation or smearing scale in addition to how divergences arise at the coincidence limit. Whether certain cross terms containing divergences have physical meaning is a question raised by the recent studies of Wu and Ford [35]. We can use these calculations to examine these issues and ask the broader question of what exactly regularization means and entails, where divergences arise and why they need to be, and not just how they ought to be treated.

Recall the smeared energy density fluctuations for the Casimir topology has the form

$$\Delta\rho_L^2(\sigma) = \Delta\rho_L^{\text{div}} + \Delta\rho_L^{\text{cross}} + \Delta\rho_L^{\text{fin}} \quad (4.36)$$

with

$$\Delta\rho_L^{\text{div}} = \chi_d (\rho_L^{\text{div}})^2 = \chi_d (\rho(\sigma))^2 \quad (4.37a)$$

$$\Delta\rho_L^{\text{cross}} = 2\chi_d \rho_L^{\text{div}} \rho_L^{\text{fin}} \quad (4.37b)$$

$$\Delta\rho_L^{\text{fin}} = 2\chi_{d,L} (\rho_L^{\text{fin}})^2 + \text{terms that vanish as } \sigma \rightarrow 0 \quad (4.37c)$$

where  $\chi_d$  is the ratio between the fluctuations for Minkowski space and the square of the corresponding energy density:  $\Delta\rho^2 = \chi_d (\rho(\sigma))^2$ . Our results show that  $\Delta\rho_L^2(\sigma)$  diverges as the width  $\sigma$  of the smearing function shrinks to zero with contributions from the truly divergent and the cross terms. We also note that the divergent term  $\Delta\rho^{\text{div}}$  is state independent, in the sense that it is independent of  $L$ , while the cross term  $\Delta\rho^{\text{cross}}$  is state dependent, as is the finite term  $\Delta\rho^{\text{fin}}$ .

If we want to ask about the strength of fluctuations of the energy density, the relevant quantity to study is the energy density correlation function  $H(x, y) = \langle \hat{\rho}(x) \hat{\rho}(y) \rangle - \langle \hat{\rho}(x) \rangle \langle \hat{\rho}(y) \rangle$ . It is finite at  $x \neq y$  for a linear quantum theory (this happens since the divergences for  $\langle \hat{\rho}(x) \hat{\rho}(y) \rangle$  are exactly the same as the product  $\langle \hat{\rho}(x) \rangle \langle \hat{\rho}(y) \rangle$ ), but diverges as  $y \rightarrow x$ , corresponding to the coincident or unsmeared limit  $\sigma \rightarrow 0$ .

To define a procedure for rendering  $\Delta\rho_L^2(\sigma)$  finite, one can see that there exists choices – which means ambiguities in the regularization scheme. Three possibilities present themselves: The first is to just drop the state independent  $\Delta\rho^{\text{div}}$ . This is easily seen to fail since we are left with the divergences from the cross term. The second is to neglect all terms that diverge as  $\sigma \rightarrow 0$ . This is too rash a move since  $\Delta\rho^{\text{cross}}$  has, along with its divergent parts, ones that are finite in the  $\sigma \rightarrow 0$  limit. This comes about since it is of the form  $\rho_L^{\text{div}} \rho_L^{\text{fin}}$  and the negative powers of  $\sigma$  present in  $\rho_L^{\text{div}}$  will cancel out against the positive powers in  $\rho_L^{\text{fin}}$ . Besides, they yield results in disagreement with earlier results using well-tested methods such as normal ordering in flat space [61] and zeta-function regularization in curved space [62].

The third choice is the one adopted by PH [8]. For the energy density, we can think of regularization as computing the contribution “above and beyond” the Minkowski vacuum contribution, same for regularizing the fluctuations. So we need to first determine for Minkowski space vacuum how the fluctuations of the energy density are related to the vacuum energy density  $\Delta\rho^2 = \chi (\rho)^2$ . This we obtained for finite smearing. For Casimir topology the sum of the divergent and cross terms take the form

$$\Delta\rho_L^{\text{div}} + \Delta\rho_L^{\text{cross}} = \chi \left\{ (\rho_L^{\text{div}})^2 + 2\rho_L^{\text{div}} \rho_L^{\text{fin}} \right\} = \chi \left\{ (\rho_L)^2 - (\rho_L^{\text{fin}})^2 \right\} \quad (4.38)$$

where  $\chi$  is the ratio derived for Minkowski vacuum. We take this to represent the (state dependent) vacuum contribution. What we find interesting is that to regularize the smeared energy density fluctuations, a state dependent subtraction must be used. With this, just the  $\sigma \rightarrow 0$  limit of the finite part  $\Delta\rho_L^{\text{fin}}$  is identified as the regularized fluctuations  $\Delta\rho_{L,\text{Reg}}^2$ . The ratio  $\chi_L$  thus obtained gives exactly the same result as derived by Kuo and Ford for  $d = 3$  via normal ordering [61] and by PH for arbitrary  $d$  via the  $\zeta$ -function [62].

That this procedure is the one to follow can be seen by considering the problem from the point separation method, see section V. For this method, the energy density expectation value is defined as the  $x' \rightarrow x$  limit of

$$\rho(x, x') = \mathcal{D}_{x,x'} G(x, x') \quad (4.39)$$

for the suitable Green function  $G(x, x')$  and  $\mathcal{D}_{x,x'}$  is a second order differential operator. In the limit  $x' \rightarrow x$ ,  $G(x, x')$  is divergent. The Green function is regularized by subtracting from it a Hadamard form  $G^L(x, x')$ :  $G_{\text{Reg}}(x, x') = G(x, x') - G^L(x, x')$  [109]. With this, the regularized energy density can be obtained

$$\rho_{\text{Reg}}(x) = \lim_{x' \rightarrow x} (\mathcal{D}_{x,x'} G_{\text{Reg}}(x, x')) \quad (4.40)$$

Or, upon re-arranging terms, define the divergent and finite pieces as

$$G^{\text{div}}(x, x') = G^L(x, x'), \quad G^{\text{fin}}(x, x') = G_{\text{Reg}}(x, x') = G(x, x') - G^L(x, x') \quad (4.41)$$

and

$$\rho(x, x') = \rho^{\text{div}}(x, x') + \rho^{\text{fin}}(x, x') \quad (4.42)$$

$$\rho^{\text{div}}(x, x') = \mathcal{D}_{x,x'} G^{\text{div}}(x, x') \quad \text{and} \quad \rho^{\text{fin}}(x, x') = \mathcal{D}_{x,x'} G^{\text{fin}}(x, x')$$

so that  $\rho_{\text{Reg}}(x) = \lim_{x' \rightarrow x} \rho^{\text{fin}}(x, x')$ , which corresponds to the  $\sigma \rightarrow 0$  limit in PH's computation of the Casimir energy density.

Now turning to the fluctuations, we have the point-separated expression for the correlation function

$$H(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{D}_{x, x'} \mathcal{D}_{y, y'} G(x, x', y, y') \quad (4.43)$$

where  $G(x, x', y, y')$  is the suitable four point function. For linear theories we can use Wick's Theorem to express this in terms of products of Green functions  $G(x, x', y, y') = G(x, y)G(x', y') + \text{permutations of } (x, x', y, y')$ . Excluded from the permutations is  $G(x, x')G(y, y')$ . (Details are in [104, 9], which includes correct identifications of needed permutations and Green functions.) The general form is

$$H(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{D}_{x, x'} \mathcal{D}_{y, y'} G(x, y) G(x', y') + \text{permutations} \quad (4.44)$$

The  $(x', y') \rightarrow (x, y)$  limits are only retained to keep track of which derivatives act on which Green functions, but we can see there are no divergences for  $y \neq x$ . However, to get the point-wise fluctuations of the energy density, the divergences from  $\lim_{y \rightarrow x} G(x, y)$  will present a problem. Splitting the Green function into its finite and divergent pieces, we can recognize terms leading to those we found for  $\Delta \rho_L^2(\sigma)$ :

$$H(x, y) = H^{\text{div}}(x, y) + H^{\text{cross}}(x, y) + H^{\text{fin}}(x, y) \quad (4.45)$$

where

$$H^{\text{div}}(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{D}_{x, x'} \mathcal{D}_{y, y'} G^{\text{div}}(x, y) G^{\text{div}}(x', y') \quad (4.46a)$$

$$H^{\text{cross}}(x, y) = 2 \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{D}_{x, x'} \mathcal{D}_{y, y'} G^{\text{div}}(x, y) G^{\text{fin}}(x', y') \quad (4.46b)$$

$$H^{\text{fin}}(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{D}_{x, x'} \mathcal{D}_{y, y'} G^{\text{fin}}(x, y) G^{\text{fin}}(x', y'), \quad (4.46c)$$

plus permutations. Thus we see the origin of both the divergent and cross terms. When the un-regularized Green function is used, we must get a cross term, along with the expected divergent term. If the fluctuations of the energy density is regularized via point separation, i.e.  $G(x, x')$  is replaced by  $G_{\text{Reg}}(x, x') = G^{\text{fin}}(x, y)$ , then we should do the same replacement for the fluctuations. When this is done, it is only the finite part above that will be left and we can define the point-wise fluctuations as

$$\Delta \rho_{\text{Reg}}^2 = \lim_{y \rightarrow x} H^{\text{fin}}(x, y) \quad (4.47)$$

The parallel with the smeared-field derivation presented here can be seen when the analysis of  $G_L(\sigma)_{,zz}$  and  $G_L(\sigma)_{,x_\perp x_\perp}$  (given in the Appendix of [8]) is considered. There it is shown they are derivatives of Green functions and can be separated into state-independent divergent part and state-dependent finite contribution:  $G_L(\sigma)_{,i} = G_{L,i}^{\text{div}} + G_{L,i}^{\text{fin}}$ , same as the split hereby shown for the Green function.

When analyzing the energy density fluctuations, discarding the divergent piece is the same as subtracting from the Green function its divergent part. If this is done, we also no longer have the cross term, just as viewing the problem from the point separation method outlined above. We feel this makes it problematic to analyze the cross term without also including the divergent term. At the same time, regularization of the fluctuations involving the subtraction of state dependent terms as realized in this calculation raises new issues on regularization which merits further investigations. In a recent work Wu and Ford [110] showed a connection between the cross term and radiation pressure.

## V. NOISE KERNEL, STRESS-ENERGY BI-TENSOR AND POINT SEPARATION

As pointed out by one of us before [3] the stress energy bi-tensor could be the starting point for a new quantum field theory constructed on spacetimes with extended structures. But for comparison with ordinary phenomena at low energy we need to find out if it behaves normally in the limit of ordinary (point-defined) quantum field theory. The method of point-separation is best suited for this purpose. The task is to seek a regularized noise-kernel for quantum fields in general curved spacetimes upon taking the coincidence limit. This was carried out by Phillips and Hu [9]. The following is a summary of their work.

PH began with a discussion of the procedures for dealing with the quantum stress tensor bi-operator at two separated points and the noise kernel and end with a general expression for the noise kernel in terms of the quantum field's Green function and its covariant derivatives up to the fourth order. (The stress tensor involves up to two covariant derivatives.) This result holds for  $x \neq y$  without recourse to renormalization of the Green function, showing that  $N_{abc'd'}(x, y)$  is always finite for  $x \neq y$  (and off the light cone for massless theories). In particular for a massless conformally coupled free scalar field on a four dimensional manifold they computed the trace of the noise kernel at both points and found this double trace vanishes identitically. This implies that there is no stochastic correction to the trace anomaly for massless conformal fields, in agreement with results arrived at in Refs. [31,73,4].

Now to obtain the point-defined quantities, one needs to deal with the divergences in the coincidence limit. For this PH adopted the “modified” point separation scheme [109,111,112] to get a regularized Green function. In this procedure, the naive Green function is rendered finite by assuming the divergences present for  $y \rightarrow x$  are state independent and can be removed by subtraction of a Hadamard form. They showed that the noise kernel in the  $y \rightarrow x$  limit is meaningful for an arbitrary curved spacetime by explicitly deriving a general expression for the noise kernel and its coincident form.

After following these expositions of PH, we will end our discussion with a reflection on the issues related to regularization we started addressing in the previous section, reiterating the important role the point-separated quantities can play in a new approach to a quantum theory of gravity. There is a fundamental shift of viewpoint in the nature and meaning of the point separation scheme: in the 70's it was used as a technique (many practitioners may still view it as a trick, even a clumsy one) for the purpose of identifying the ultraviolet divergences. Now in the new approach (as advocated by one of us [3,55]) we want to use the point-separated expressions to construct a quantum field theory for extended spacetimes.

## A. Point Separation

The point separation scheme introduced in the 60's by DeWitt [113] was brought to more popular use in the 70's in the context of quantum field theory in curved spacetimes [114,58] as a means for obtaining a finite quantum stress tensor<sup>3</sup>. Since the stress-energy tensor is built from the product of a pair of field operators evaluated at a single point, it is not well-defined. In this scheme, one introduces an artificial separation of the single point  $x$  to a pair of closely separated points  $x$  and  $x'$ . The problematic terms involving field products such as  $\hat{\phi}(x)^2$  becomes  $\hat{\phi}(x)\hat{\phi}(x')$ , whose expectation value is well defined. If one is interested in the low energy behavior captured by the point-defined quantum field theory – as the effort in the 70's was directed – one takes the coincidence limit. Once the divergences present are identified, they may be removed (regularization) or moved (by renormalizing the coupling constants), to produce a well-defined, finite stress tensor at a single point.

Thus the first order of business is the construction of the stress tensor and then derive the symmetric stress-energy tensor two point function, the noise kernel, in terms of the Wightman Green function. In this section we will use the traditional notation for index tensors in the point-separation context.

### 1. $n$ -tensors and end-point expansions

An object like the Green function  $G(x, y)$  is an example of a *bi-scalar*: it transforms as scalar at both points  $x$  and  $y$ . We can also define a *bi-tensor*  $T_{a_1 \dots a_n b'_1 \dots b'_m}(x, y)$ : upon a coordinate transformation, this transforms as a rank  $n$  tensor at  $x$  and a rank  $m$  tensor at  $y$ . We will extend this up to a *quad-tensor*  $T_{a_1 \dots a_{n_1} b'_1 \dots b'_{n_2} c''_1 \dots c''_{n_3} d'''_1 \dots d'''_{n_4}}$  which has support at four points  $x, y, x', y'$ , transforming as rank  $n_1, n_2, n_3, n_4$  tensors at each of the four points. This also sets the notation we will use: unprimed indices referring to the tangent space constructed above  $x$ , single primed indices to  $y$ , double primed to  $x'$  and triple primed to  $y'$ . For each point, there is the covariant derivative  $\nabla_a$  at that

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<sup>3</sup>We know there are several regularization methods developed for the removal of ultraviolet divergences in the stress energy tensor of quantum fields in curved spacetime [20–22]. Their mutual relations are known, and discrepancies explained. This formal structure of regularization schemes for quantum fields in curved spacetime should remain intact as we apply them to the regularization of the noise kernel in general curved spacetimes. Specific considerations will of course enter for each method. But for the methods employed so far, such as zeta-function, point separation, dimensional, smeared-field [62,9] applied to simple cases (Casimir, Einstein, thermal fields) there is no new inconsistency or discrepancy.

point. Covariant derivatives at different points commute and the covariant derivative at, say, point  $x'$ , does not act on a bi-tensor defined at, say,  $x$  and  $y$ :

$$T_{ab';c;d'} = T_{ab';d';c} \quad \text{and} \quad T_{ab';c''} = 0. \quad (5.1)$$

To simplify notation, henceforth we will eliminate the semicolons after the first one for multiple covariant derivatives at multiple points.

Having objects defined at different points, the coincident limit is defined as evaluation “on the diagonal”, in the sense of the spacetime support of the function or tensor, and the usual shorthand  $[G(x, y)] \equiv G(x, x)$  is used. This extends to  $n$ -tensors as

$$\left[ T_{a_1 \dots a_{n_1} b'_1 \dots b'_{n_2} c''_1 \dots c''_{n_3} d'''_1 \dots d'''_{n_4}} \right] = T_{a_1 \dots a_{n_1} b_1 \dots b_{n_2} c_1 \dots c_{n_3} d_1 \dots d_{n_4}}, \quad (5.2)$$

*i.e.*, this becomes a rank  $(n_1 + n_2 + n_3 + n_4)$  tensor at  $x$ . The multi-variable chain rule relates covariant derivatives acting at different points, when we are interested in the coincident limit:

$$[T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c} = [T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c} + [T_{a_1 \dots a_m b'_1 \dots b'_n}]_{;c'}. \quad (5.3)$$

This result is referred to as *Synge's theorem* in this context; we follow Fulling's [21] discussion.

The bi-tensor of *parallel transport*  $g_a^{b'}$  is defined such that when it acts on a vector  $v_{b'}$  at  $y$ , it parallel transports the vector along the geodesics connecting  $x$  and  $y$ . This allows us to add vectors and tensors defined at different points. We cannot directly add a vector  $v_a$  at  $x$  and vector  $w_{a'}$  at  $y$ . But by using  $g_a^{b'}$ , we can construct the sum  $v^a + g_a^{b'} w_{b'}$ . We will also need the obvious property  $[g_a^{b'}] = g_a^{b'}$ .

The main bi-scalar we need is the *world function*  $\sigma(x, y)$ . This is defined as a half of the square of the geodesic distance between the points  $x$  and  $y$ . It satisfies the equation

$$\sigma = \frac{1}{2} \sigma^{;p} \sigma_{;p} \quad (5.4)$$

Often in the literature, a covariant derivative is implied when the world function appears with indices:  $\sigma^a \equiv \sigma^{;a}$ , *i.e.* taking the covariant derivative at  $x$ , while  $\sigma^{a'}$  means the covariant derivative at  $y$ . This is done since the vector  $-\sigma^a$  is the tangent vector to the geodesic with length equal the distance between  $x$  and  $y$ . As  $\sigma^a$  records information about distance and direction for the two points this makes it ideal for constructing a series expansion of a bi-scalar. The *end point* expansion of a bi-scalar  $S(x, y)$  is of the form

$$S(x, y) = A^{(0)} + \sigma^p A_p^{(1)} + \sigma^p \sigma^q A_{pq}^{(2)} + \sigma^p \sigma^q \sigma^r A_{pqr}^{(3)} + \sigma^p \sigma^q \sigma^r \sigma^s A_{pqrs}^{(4)} + \dots \quad (5.5)$$

where, following our convention, the expansion tensors  $A_{a_1 \dots a_n}^{(n)}$  with unprimed indices have support at  $x$  and hence the name end point expansion. Only the symmetric part of these tensors contribute to the expansion. For the purposes of multiplying series expansions it is convenient to separate the distance dependence from the direction dependence. This is done by introducing the unit vector  $p^a = \sigma^a / \sqrt{2\sigma}$ . Then the series expansion can be written

$$S(x, y) = A^{(0)} + \sigma^{\frac{1}{2}} A^{(1)} + \sigma A^{(2)} + \sigma^{\frac{3}{2}} A^{(3)} + \sigma^2 A^{(4)} + \dots \quad (5.6)$$

The expansion scalars are related to the expansion tensors via  $A^{(n)} = 2^{n/2} A_{p_1 \dots p_n}^{(n)} p^{p_1} \dots p^{p_n}$ .

The last object we need is the *Van Vleck-Morette* determinant  $D(x, y)$ , defined as  $D(x, y) \equiv -\det(-\sigma_{;ab'})$ . The related bi-scalar

$$\Delta^{1/2} = \left( \frac{D(x, y)}{\sqrt{g(x)g(y)}} \right)^{\frac{1}{2}} \quad (5.7)$$

satisfies the equation

$$\Delta^{1/2} (4 - \sigma_{;p}^{;p}) - 2\Delta^{1/2}_{;p} \sigma^{;p} = 0 \quad (5.8)$$

with the boundary condition  $[\Delta^{1/2}] = 1$ .

Further details on these objects and discussions of the definitions and properties are contained in [58] and [115]. There it is shown how the defining equations for  $\sigma$  and  $\Delta^{1/2}$  are used to determine the coincident limit expression for the various covariant derivatives of the world function ( $[\sigma_{;a}]$ ,  $[\sigma_{;ab}]$ , *etc.*) and how the defining differential equation for  $\Delta^{1/2}$  can be used to determine the series expansion of  $\Delta^{1/2}$ . We show how the expansion tensors  $A_{a_1 \dots a_n}^{(n)}$  are determined in terms of the coincident limits of covariant derivatives of the bi-scalar  $S(x, y)$ . (Ref. [115] details how point separation can be implemented on the computer to provide easy access to a wider range of applications involving higher derivatives of the curvature tensors.)

## B. Stress Energy Bi-Tensor Operator and Noise Kernel

Even though we believe that the point-separated results are more basic in the sense that it reflects a deeper structure of the quantum theory of spacetime, we will nevertheless start with quantities defined at one point because they are more familiar in conventional quantum field theory. We will use point separation to introduce the biquantities. The key issue here is thus the distinction between point-defined (*pt*) and point-separated (*bi*) quantities.

For a free classical scalar field  $\phi$  with the action  $S_m[g, \phi]$  defined in Eq. (2.1), the classical stress-energy tensor is

$$T_{ab} = (1 - 2\xi) \phi_{;a} \phi_{;b} + \left(2\xi - \frac{1}{2}\right) \phi_{;p} \phi^{;p} g_{ab} + 2\xi \phi (\phi_{;p}{}^p - \phi_{;ab} g_{ab}) + \phi^2 \xi \left(R_{ab} - \frac{1}{2} R g_{ab}\right) - \frac{1}{2} m^2 \phi^2 g_{ab}, \quad (5.9)$$

which is equivalent to the tensor of Eq. (2.3) but written in a slightly different form for convenience. When we make the transition to quantum field theory, we promote the field  $\phi(x)$  to a field operator  $\hat{\phi}(x)$ . The fundamental problem of defining a quantum operator for the stress tensor is immediately visible: the field operator appears quadratically. Since  $\hat{\phi}(x)$  is an operator-valued distribution, products at a single point are not well-defined. But if the product is point separated ( $\hat{\phi}^2(x) \rightarrow \hat{\phi}(x)\hat{\phi}(x')$ ), they are finite and well-defined.

Let us first seek a point-separated extension of these classical quantities and then consider the quantum field operators. Point separation is symmetrically extended to products of covariant derivatives of the field according to

$$(\phi_{;a})(\phi_{;b}) \rightarrow \frac{1}{2} \left( g_a{}^{p'} \nabla_{p'} \nabla_b + g_b{}^{p'} \nabla_a \nabla_{p'} \right) \phi(x) \phi(x'), \quad (5.10)$$

$$\phi (\phi_{;ab}) \rightarrow \frac{1}{2} \left( \nabla_a \nabla_b + g_a{}^{p'} g_b{}^{q'} \nabla_{p'} \nabla_{q'} \right) \phi(x) \phi(x'). \quad (5.11)$$

The bi-vector of parallel displacement  $g_a{}^{a'}(x, x')$  is included so that we may have objects that are rank 2 tensors at  $x$  and scalars at  $x'$ .

To carry out point separation on (5.9), we first define the differential operator

$$\begin{aligned} \mathcal{T}_{ab} = & \frac{1}{2} (1 - 2\xi) \left( g_a{}^{a'} \nabla_{a'} \nabla_b + g_b{}^{b'} \nabla_a \nabla_{b'} \right) + \left( 2\xi - \frac{1}{2} \right) g_{ab} g^{cd'} \nabla_c \nabla_{d'} \\ & - \xi \left( \nabla_a \nabla_b + g_a{}^{a'} g_b{}^{b'} \nabla_{a'} \nabla_{b'} \right) + \xi g_{ab} \left( \nabla_c \nabla^c + \nabla_{c'} \nabla^{c'} \right) \\ & + \xi \left( R_{ab} - \frac{1}{2} g_{ab} R \right) - \frac{1}{2} m^2 g_{ab} \end{aligned} \quad (5.12)$$

from which we obtain the classical stress tensor as

$$T_{ab}(x) = \lim_{x' \rightarrow x} \mathcal{T}_{ab} \phi(x) \phi(x'). \quad (5.13)$$

That the classical tensor field no longer appears as a product of scalar fields at a single point allows a smooth transition to the quantum tensor field. From the viewpoint of the stress tensor, the separation of points is an artificial construct so when promoting the classical field to a quantum one, neither point should be favored. The product of field configurations is taken to be the symmetrized operator product, denoted by curly brackets:

$$\phi(x) \phi(y) \rightarrow \frac{1}{2} \left\{ \hat{\phi}(x), \hat{\phi}(y) \right\} = \frac{1}{2} \left( \hat{\phi}(x) \hat{\phi}(y) + \hat{\phi}(y) \hat{\phi}(x) \right) \quad (5.14)$$

With this, the point separated stress energy tensor operator is defined as

$$\hat{T}_{ab}(x, x') \equiv \frac{1}{2} \mathcal{T}_{ab} \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\}. \quad (5.15)$$

While the classical stress tensor was defined at the coincidence limit  $x' \rightarrow x$ , we cannot attach any physical meaning to the quantum stress tensor at one point until the issue of regularization is dealt with, which will happen in the next section. For now, we will maintain point separation so as to have a mathematically meaningful operator.

The expectation value of the point-separated stress tensor can now be taken. This amounts to replacing the field operators by their expectation value, which is given by the Hadamard (or Schwinger) function



$$G^{(1)}(x, x') = \langle \{ \hat{\phi}(x), \hat{\phi}(x') \} \rangle. \quad (5.16)$$

and the point-separated stress tensor is defined as

$$\langle \hat{T}_{ab}(x, x') \rangle = \frac{1}{2} \mathcal{T}_{ab} G^{(1)}(x, x') \quad (5.17)$$

where, since  $\mathcal{T}_{ab}$  is a differential operator, it can be taken “outside” the expectation value. The expectation value of the point-separated quantum stress tensor for a free, massless ( $m = 0$ ) conformally coupled ( $\xi = 1/6$ ) scalar field on a four dimension spacetime with scalar curvature  $R$  is

$$\begin{aligned} \langle \hat{T}_{ab}(x, x') \rangle = & \frac{1}{6} \left( g^{p'}_b G^{(1)}_{;p'a} + g^{p'}_a G^{(1)}_{;p'b} \right) - \frac{1}{12} g^{p'}_q G^{(1)}_{;p'q} g_{ab} \\ & - \frac{1}{12} \left( g^{p'}_a g^{q'}_b G^{(1)}_{;p'q'} + G^{(1)}_{;ab} \right) + \frac{1}{12} \left( \left( G^{(1)}_{;p'p'} + G^{(1)}_{;p'p} \right) g_{ab} \right) \\ & + \frac{1}{12} G^{(1)} \left( R_{ab} - \frac{1}{2} R g_{ab} \right) \end{aligned} \quad (5.18)$$

### 1. Finiteness of Noise Kernel

We now turn our attention to the noise kernel introduced in Eq. (2.11), which is the symmetrized product of the (mean subtracted) stress tensor operator:

$$\begin{aligned} 8N_{ab,c'd'}(x, y) &= \langle \{ \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle, \hat{T}_{c'd'}(y) - \langle \hat{T}_{c'd'}(y) \rangle \} \rangle \\ &= \langle \{ \hat{T}_{ab}(x), \hat{T}_{c'd'}(y) \} \rangle - 2\langle \hat{T}_{ab}(x) \rangle \langle \hat{T}_{c'd'}(y) \rangle \end{aligned} \quad (5.19)$$

Since  $\hat{T}_{ab}(x)$  defined at one point can be ill-behaved as it is generally divergent, one can question the soundness of these quantities. But as will be shown later, the noise kernel is finite for  $y \neq x$ . All field operator products present in the first expectation value that could be divergent are canceled by similar products in the second term. We will replace each of the stress tensor operators in the above expression for the noise kernel by their point separated versions, effectively separating the two points  $(x, y)$  into the four points  $(x, x', y, y')$ . This will allow us to express the noise kernel in terms of a pair of differential operators acting on a combination of four and two point functions. Wick’s theorem will allow the four point functions to be re-expressed in terms of two point functions. From this we see that all possible divergences for  $y \neq x$  will cancel. When the coincidence limit is taken divergences do occur. The above procedure will allow us to isolate the divergences and obtain a finite result.

Taking the point-separated quantities as more basic, one should replace each of the stress tensor operators in the above with the corresponding point separated version (5.15), with  $\mathcal{T}_{ab}$  acting at  $x$  and  $x'$  and  $\mathcal{T}_{c'd'}$  acting at  $y$  and  $y'$ . In this framework the noise kernel is defined as

$$8N_{ab,c'd'}(x, y) = \lim_{x' \rightarrow x} \lim_{y' \rightarrow y} \mathcal{T}_{ab} \mathcal{T}_{c'd'} G(x, x', y, y') \quad (5.20)$$

where the four point function is

$$\begin{aligned} G(x, x', y, y') &= \frac{1}{4} \left[ \langle \{ \{ \hat{\phi}(x), \hat{\phi}(x') \}, \{ \hat{\phi}(y), \hat{\phi}(y') \} \} \rangle \right. \\ &\quad \left. - 2 \langle \{ \hat{\phi}(x), \hat{\phi}(x') \} \rangle \langle \{ \hat{\phi}(y), \hat{\phi}(y') \} \rangle \right]. \end{aligned} \quad (5.21)$$

We assume the pairs  $(x, x')$  and  $(y, y')$  are each within their respective Riemann normal coordinate neighborhoods so as to avoid problems that possible geodesic caustics might be present. When we later turn our attention to computing the limit  $y \rightarrow x$ , after issues of regularization are addressed, we will want to assume all four points are within the same Riemann normal coordinate neighborhood.

Wick’s theorem, for the case of free fields which we are considering, gives the simple product four point function in terms of a sum of products of Wightman functions (we use the shorthand notation  $G_{xy} \equiv G_+(x, y) = \langle \hat{\phi}(x) \hat{\phi}(y) \rangle$ ):

$$\langle \hat{\phi}(x) \hat{\phi}(y) \hat{\phi}(x') \hat{\phi}(y') \rangle = G_{xy'} G_{yx'} + G_{xx'} G_{yy'} + G_{xy} G_{x'y'} \quad (5.22)$$

Expanding out the anti-commutators in (5.21) and applying Wick's theorem, the four point function becomes

$$G(x, x', y, y') = G_{xy'} G_{x'y} + G_{xy} G_{x'y'} + G_{yx'} G_{y'x} + G_{yx} G_{y'x'} \quad (5.23)$$

We can now easily see that the noise kernel defined via this function is indeed well defined for the limit  $(x', y') \rightarrow (x, y)$ :

$$G(x, x, y, y) = 2 (G_{xy}^2 + G_{yx}^2). \quad (5.24)$$

From this we can see that the noise kernel is also well defined for  $y \neq x$ ; any divergence present in the first expectation value of (5.21) have been cancelled by those present in the pair of Green functions in the second term, in agreement with the results of section II.

## 2. Explicit Form of the Noise Kernel

We will let the points separated for a while so we can keep track of which covariant derivative acts on which arguments of which Wightman function. As an example (the complete calculation is quite long), consider the result of the first set of covariant derivative operators in the differential operator (5.12), from both  $\mathcal{T}_{ab}$  and  $\mathcal{T}_{c'd'}$ , acting on  $G(x, x', y, y')$ :

$$\begin{aligned} & \frac{1}{4} (1 - 2\xi)^2 \left( g_a^{p''} \nabla_{p''} \nabla_b + g_b^{p''} \nabla_{p''} \nabla_a \right) \\ & \times \left( g_c^{q'''} \nabla_{q'''} \nabla_{d'} + g_{d'}^{q'''} \nabla_{q'''} \nabla_c \right) G(x, x', y, y') \end{aligned} \quad (5.25)$$

(Our notation is that  $\nabla_a$  acts at  $x$ ,  $\nabla_{c'}$  at  $y$ ,  $\nabla_{b''}$  at  $x'$  and  $\nabla_{d''}$  at  $y'$ ). Expanding out the differential operator above, we can determine which derivatives act on which Wightman function:

$$\begin{aligned} & \frac{(1 - 2\xi)^2}{4} \times \left[ g_c^{p'''} g^{q''}{}_a (G_{xy';bp'''} G_{x'y;q''d'} + G_{xy;bd'} G_{x'y';q''p'''} \right. \\ & \quad + G_{yx';q''d'} G_{y'x;bp'''} + G_{yx;bd'} G_{y'x';q''p'''} \\ & \quad + g_{d'}^{p'''} g^{q''}{}_a (G_{xy';bp'''} G_{x'y;q''c'} + G_{xy;bc'} G_{x'y';q''p'''} \\ & \quad + G_{yx';q''c'} G_{y'x;bp'''} + G_{yx;bc'} G_{y'x';q''p'''} \\ & \quad + g_c^{p'''} g^{q''}{}_b (G_{xy';ap'''} G_{x'y;q''d'} + G_{xy;ad'} G_{x'y';q''p'''} \\ & \quad + G_{yx';q''d'} G_{y'x;ap'''} + G_{yx;ad'} G_{y'x';q''p'''} \\ & \quad \left. + g_{d'}^{p'''} g^{q''}{}_b (G_{xy';ap'''} G_{x'y;q''c'} + G_{xy;ac'} G_{x'y';q''p'''} \right. \\ & \quad \left. + G_{yx';q''c'} G_{y'x;ap'''} + G_{yx;ac'} G_{y'x';q''p'''} \right] \end{aligned} \quad (5.26)$$

If we now let  $x' \rightarrow x$  and  $y' \rightarrow y$  the contribution to the noise kernel is (including the factor of  $\frac{1}{8}$  present in the definition of the noise kernel):

$$\begin{aligned} & \frac{1}{8} \left\{ (1 - 2\xi)^2 (G_{xy;ad'} G_{xy;bc'} + G_{xy;ac'} G_{xy;bd'}) \right. \\ & \quad \left. + (1 - 2\xi)^2 (G_{yx;ad'} G_{yx;bc'} + G_{yx;ac'} G_{yx;bd'}) \right\} \end{aligned} \quad (5.27)$$

That this term can be written as the sum of a part involving  $G_{xy}$  and one involving  $G_{yx}$  is a general property of the entire noise kernel. It thus takes the form

$$N_{abc'd'}(x, y) = N_{abc'd'}[G_+(x, y)] + N_{abc'd'}[G_+(y, x)]. \quad (5.28)$$

We will present the form of the functional  $N_{abc'd'}[G]$  shortly. First we note, for  $x$  and  $y$  time-like separated, the above split of the noise kernel allows us to express it in terms of the Feynman (time ordered) Green function  $G_F(x, y)$  and the Dyson (anti-time ordered) Green function  $G_D(x, y)$ :

$$N_{abc'd'}(x, y) = N_{abc'd'}[G_F(x, y)] + N_{abc'd'}[G_D(x, y)] \quad (5.29)$$

<sup>4</sup> The complete form of the functional  $N_{abc'd'}[G]$  is

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<sup>4</sup>This can be connected with the zeta function approach to this problem [62] as follows: Recall when the quantum stress tensor fluctuations determined in the Euclidean section is analytically continued back to Lorentzian signature ( $\tau \rightarrow it$ ), the

$$N_{abc'd'} [G] = \tilde{N}_{abc'd'} [G] + g_{ab} \tilde{N}_{c'd'} [G] + g_{c'd'} \tilde{N}'_{ab} [G] + g_{ab} g_{c'd'} \tilde{N} [G] \quad (5.31a)$$

with

$$\begin{aligned} 8\tilde{N}_{abc'd'} [G] = & (1 - 2\xi)^2 (G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b}) + 4\xi^2 (G_{;c'd'} G_{;ab} + G_{;ab} G_{;c'd'}) \\ & - 2\xi (1 - 2\xi) (G_{;b} G_{;c'a d'} + G_{;a} G_{;c'b d'} + G_{;d'} G_{;abc'} + G_{;c'} G_{;abd'}) \\ & + 2\xi (1 - 2\xi) (G_{;a} G_{;b} R_{c'd'} + G_{;c'} G_{;d'} R_{ab}) \\ & - 4\xi^2 (G_{;ab} R_{c'd'} + G_{;c'd'} R_{ab}) G + 2\xi^2 R_{c'd'} R_{ab} G^2 \end{aligned} \quad (5.31b)$$

$$\begin{aligned} 8\tilde{N}'_{ab} [G] = & 2(1 - 2\xi) \left( \left( 2\xi - \frac{1}{2} \right) G_{;p'b} G_{;p'a} + \xi (G_{;b} G_{;p'a p'} + G_{;a} G_{;p'b p'}) \right) \\ & - 4\xi \left( \left( 2\xi - \frac{1}{2} \right) G_{;p'} G_{;ab p'} + \xi (G_{;p'} G_{;ab} + G_{;ab} G_{;p' p'}) \right) \\ & - (m^2 + \xi R') ((1 - 2\xi) G_{;a} G_{;b} - 2G\xi G_{;ab}) \\ & + 2\xi \left( \left( 2\xi - \frac{1}{2} \right) G_{;p'} G_{;p'} + 2G\xi G_{;p' p'} \right) R_{ab} \\ & - (m^2 + \xi R') \xi R_{ab} G^2 \end{aligned} \quad (5.31c)$$

$$\begin{aligned} 8\tilde{N} [G] = & 2 \left( 2\xi - \frac{1}{2} \right)^2 G_{;p'q} G_{;p'q} + 4\xi^2 (G_{;p'p'} G_{;q^q} + G_{;p^p} G_{;q'q'}) \\ & + 4\xi \left( 2\xi - \frac{1}{2} \right) (G_{;p} G_{;q'p'q'} + G_{;p'} G_{;q^q p'}) \\ & - \left( 2\xi - \frac{1}{2} \right) ((m^2 + \xi R) G_{;p'} G_{;p'} + (m^2 + \xi R') G_{;p} G_{;p}) \\ & - 2\xi ((m^2 + \xi R) G_{;p'p'} + (m^2 + \xi R') G_{;p^p}) G \\ & + \frac{1}{2} (m^2 + \xi R) (m^2 + \xi R') G^2 \end{aligned} \quad (5.31d)$$

### 3. Trace of the Noise Kernel

One of the most interesting and surprising results to come out of the investigations undertaken in the 1970's of the quantum stress tensor was the discovery of the trace anomaly [116]. When the trace of the stress tensor  $T = g^{ab} T_{ab}$  is evaluated for a field configuration that satisfies the field equation (2.2) the trace is seen to vanish for massless conformally coupled fields. When this analysis is carried over to the renormalized expectation value of the quantum stress tensor, the trace no longer vanishes. Wald [112] showed this was due to the failure of the renormalized Hadamard function  $G_{\text{ren}}(x, x')$  to be symmetric in  $x$  and  $x'$ , implying it does not necessarily satisfy the field equation (2.2) in the variable  $x'$ . The definition of  $G_{\text{ren}}(x, x')$  in the context of point separation will come next.)

With this in mind, we can now determine the noise associated with the trace. Taking the trace at both points  $x$  and  $y$  of the noise kernel functional (5.29):

$$\begin{aligned} N [G] = & g^{ab} g^{c'd'} N_{abc'd'} [G] \\ = & -3G\xi \left\{ \left( m^2 + \frac{1}{2}\xi R \right) G_{;p'p'} + \left( m^2 + \frac{1}{2}\xi R' \right) G_{;p^p} \right\} \\ & + \frac{9\xi^2}{2} \left\{ G_{;p'p'} G_{;p^p} + G_{;p^p} G_{;p'p'} \right\} + \left( m^2 + \frac{1}{2}\xi R \right) \left( m^2 + \frac{1}{2}\xi R' \right) G^2 \end{aligned}$$

time ordered product results. On the other hand, if the continuation is  $\tau \rightarrow -it$ , the anti-time ordered product results. With this in mind, the noise kernel is seen to be related to the quantum stress tensor fluctuations derived via the effective action as

$$16N_{abc'd'} = \Delta T_{abc'd'}^2 \Big|_{t=-i\tau, t'=-i\tau'} + \Delta T_{abc'd'}^2 \Big|_{t=i\tau, t'=i\tau'} . \quad (5.30)$$

$$+3 \left( \frac{1}{6} - \xi \right) \left\{ +3 \left( \frac{1}{6} - \xi \right) G_{;p'p} G_{;{}^{p'}p} - 3\xi \left( G_{;p} G_{;p'}{}^{pp'} + G_{;p'} G_{;p}{}^{pp'} \right) \right. \\ \left. \left( m^2 + \frac{1}{2}\xi R \right) G_{;p'} G_{;{}^{p'}p} + \left( m^2 + \frac{1}{2}\xi R' \right) G_{;p} G_{;{}^p} \right\} \quad (5.32)$$

For the massless conformal case, this reduces to

$$N[G] = \frac{1}{144} \{ RR'G^2 - 6G(R\Box' + R'\Box)G + 18((\Box G)(\Box' G) + \Box'\Box G) \} \quad (5.33)$$

which holds for any function  $G(x, y)$ . For  $G$  being the Green function, it satisfies the field equation (2.2):

$$\Box G = (m^2 + \xi R)G \quad (5.34)$$

We will only assume the Green function satisfies the field equation in its first variable. Using the fact  $\Box'R = 0$  (because the covariant derivatives act at a different point than at which  $R$  is supported), it follows that

$$\Box'\Box G = (m^2 + \xi R)\Box'G. \quad (5.35)$$

With these results, the noise kernel trace becomes

$$N[G] = \frac{1}{2} \left( m^2 (1 - 3\xi) + 3R \left( \frac{1}{6} - \xi \right) \xi \right) \\ \times \left\{ G^2 (2m^2 + R'\xi) + (1 - 6\xi) G_{;p'} G_{;{}^{p'}} - 6G\xi G_{;p'}{}^{p'} \right\} \\ + \frac{1}{2} \left( \frac{1}{6} - \xi \right) \left\{ 3 (2m^2 + R'\xi) G_{;p} G_{;{}^p} - 18\xi G_{;p} G_{;p'}{}^{pp'} \right. \\ \left. + 18 \left( \frac{1}{6} - \xi \right) G_{;p'p} G_{;{}^{p'}p} \right\} \quad (5.36)$$

which vanishes for the massless conformal case. We have thus shown, based solely on the definition of the point separated noise kernel, there is no noise associated with the trace anomaly. This result obtained in Ref. [9] is completely general since it is assumed that the Green function is only satisfying the field equations in its first variable; an alternative proof of this result was given in Ref. [4]. This condition holds not just for the classical field case, but also for the regularized quantum case, where one does not expect the Green function to satisfy the field equation in both variables. One can see this result from the simple observation used in section II: since the trace anomaly is known to be locally determined and quantum state independent, whereas the noise present in the quantum field is non-local, it is hard to find a noise associated with it. This general result is in agreement with previous findings [31,30,73], derived from the Feynman-Vernon influence functional formalism [76,77] for some particular cases.

### C. Regularization of the Noise Kernel

We pointed out earlier that field quantities defined at two separated points possess important information which could be the starting point for probes into possible extended structures of spacetime. Moving in the other (homeward) direction, it is of interest to see how fluctuations of energy momentum (loosely, noise) would enter in the ordinary (point-wise) quantum field theory in helping us to address a new set of issues such as a) whether the fluctuations to mean ratio can act as a criterion for the validity of semiclassical gravity. b) Whether the fluctuations in the vacuum energy density which drives inflationary cosmology violates the positive energy condition, c) How can we derive structure formation from quantum fluctuations, or d) General relativity as a low energy effective theory in the geometro- hydrodynamic limit [53,3].

For these inquires we need to construct regularization procedures to remove the ultraviolet divergences in the coincidence limit. The goal is to obtain a finite expression for the noise kernel in this limit.

We can see from the point separated form of the stress tensor (5.17) what we need to regularize is the Green function  $G^{(1)}(x, x')$ . Once the Green function has been regularized such that it is smooth and has a well defined  $x' \rightarrow x$  limit, the stress tensor will be well defined. In Minkowski space, this issue is easily resolved by a "normal ordering" prescription, which hinges on the existence of a unique vacuum. Unfortunately, for a general curved spacetime, there is no unique vacuum, and hence, no unique mode expansion on which to build a normal ordering prescription. But we can still ask if there is a way to determine a contribution we can subtract to yield a unique quantum stress tensor.

Here we follow the prescription of Wald [109], and Adler *et. al.* [111] (with corrections [112]) summarized in [22]. We give a short synopsis below as it will be referred to in subsequent discussions.

The idea builds on the fact that for  $G(x, x')_\omega = \langle \omega | \hat{\phi}(x) \hat{\phi}(x') | \omega \rangle$ , the function

$$F(x, x') = G(x, x')_{\omega_1} - G(x, x')_{\omega_2} \quad (5.37)$$

is a smooth function of  $x$  and  $x'$ , where  $\omega_1$  and  $\omega_2$  denote two different states. This means the difference between the stress tensor for two states is well defined for the point separation scheme, *i.e.*,

$$\mathcal{F}_{ab} = \frac{1}{2} \lim_{x' \rightarrow x} \mathcal{T}_{ab} (F(x, x') + F(x', x)) \quad (5.38)$$

is well defined. So a bi-distribution  $G^L(x, x')$  might be useful for the vacuum subtraction. At first, it would seem unlikely we could find such a unique bi-distribution. Wald found that with the introduction of four axioms for the regularized stress tensor

$$\langle \hat{T}_{ab}(x) \rangle_{\text{ren}} = \lim_{x' \rightarrow x} \frac{1}{2} \mathcal{T}_{ab} (G^{(1)}(x, x') - G^L(x, x')) \quad (5.39)$$

$G^L(x, x')$  is uniquely determined, up to a local conserved curvature term. The Wald axioms are [112,22]:

1. The difference between the stress tensor for two states should agree with (5.38);
2. The stress tensor should be local with respect to the state of the field;
3. For all states, the stress tensor is conserved:  $\nabla^a \langle T_{ab} \rangle = 0$ ;
4. In Minkowski space, the result  $\langle 0 | T_{ab} | 0 \rangle = 0$  is recovered.

We are still left with the problem of determining the form of such a bi-distribution. Hadamard [117] showed that the Green function for a large class of states takes the form (in four spacetime dimensions)

$$G^L(x, x') = \frac{1}{8\pi^2} \left( \frac{2U(x, x')}{\sigma} + V(x, x') \log \sigma + W(x, x') \right) \quad (5.40)$$

with  $U(x, x')$ ,  $V(x, x')$  and  $W(x, x')$  being smooth functions<sup>5</sup>. We refer to Eqn (5.40) as the ‘‘Hadamard ansatz’’.

Since the functions  $V(x, x')$  and  $W(x, x')$  are smooth functions, they can be expanded as

$$V(x, x') = \sum_{n=0}^{\infty} v_n(x, x') \sigma^n, \quad (5.41a)$$

$$W(x, x') = \sum_{n=0}^{\infty} w_n(x, x') \sigma^n, \quad (5.41b)$$

with the  $v_n$ ’s and  $w_n$ ’s themselves smooth functions. These functions and  $U(x, x')$  are determined by substituting  $G^L(x, x')$  in the wave equation  $KG^L(x, x') = 0$  and equating to zero the coefficients of the explicitly appearing powers of  $\sigma^n$  and  $\sigma^n \log \sigma$ . Doing so, we get the infinite set of equations

$$U(x, x') = \Delta^{1/2}; \quad (5.42a)$$

$$2H_0 v_0 + K \Delta^{1/2} = 0; \quad (5.42b)$$

$$2nH_n v_n + K v_{n-1} = 0, \quad n \geq 1; \quad (5.42c)$$

$$2H_{2n} v_n + 2nH_n w_n + K w_{n-1} = 0, \quad n \geq 1 \quad (5.42d)$$

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<sup>5</sup>When working in the Lorentz sector of a field theory, *i.e.*, when the metric signature is  $(-, +, +, +)$ , as opposed to the Euclidean sector with the signature  $(+, +, +, +)$ , we must modify the above function to account for null geodesics, since  $\sigma(x, x') = 0$  for null separated  $x$  and  $x'$ . This problem can be overcome by using  $\sigma \rightarrow \sigma + 2i\epsilon(t - t') + \epsilon^2$ . Here, we will work only with geometries that possess Euclidean sectors and carry out our analysis with Riemannian geometries and only at the end continue back to the Lorentzian geometry. Nonetheless, this presents no difficulty. At any point in the analysis the above replacement for  $\sigma$  can be performed.

with

$$H_n = \sigma^{ip} \nabla_p + \left( n - 1 + \frac{1}{2} (\square \sigma) \right). \quad (5.42e)$$

From Eqs (5.42d), the functions  $v_n$  are completely determined. In fact, they are symmetric functions, and hence  $V(x, x')$  is a symmetric function of  $x$  and  $x'$ . On the other hand, the field equations only determine  $w_n$ ,  $n \geq 1$ , leaving  $w_0(x, x')$  completely arbitrary. This reflects the state dependence of the Hadamard form above. Moreover, even if  $w_0(x, x')$  is chosen to be symmetric, this does not guarantee that  $W(x, x')$  will be. By using axiom (4)  $w_0(x, x') \equiv 0$ . With this choice, the Minkowski spacetime limit is

$$G^L = \frac{1}{(2\pi)^2} \frac{1}{\sigma}, \quad (5.43)$$

where now  $2\sigma = (t - t')^2 - (\mathbf{x} - \mathbf{x}')^2$  and this corresponds to the correct vacuum contribution that needs to be subtracted.

With this choice though, we are left with a  $G^L(x, x')$  which is not symmetric and hence does not satisfy the field equation at  $x'$ , for fixed  $x$ . Wald [112] showed this in turn implies axiom (3) is not satisfied. He resolved this problem by adding to the regularized stress tensor a term which cancels that which breaks the conservation of the old stress tensor:

$$\langle T_{ab}^{\text{new}} \rangle = \langle T_{ab}^{\text{old}} \rangle + \frac{1}{2(4\pi)^2} g_{ab} [v_1], \quad (5.44)$$

where  $[v_1] = v_1(x, x)$  is the coincident limit of the  $n = 1$  solution of Eq (5.42d). This yields a stress tensor which satisfies all four axioms and produces the well known trace anomaly  $\langle T_a^a \rangle = [v_1]/8\pi^2$ . We can view this redefinition as taking place at the level of the stress tensor operator via

$$\hat{T}_{ab} \rightarrow \hat{T}_{ab} + \frac{\hat{1}}{2(4\pi)^2} g_{ab} [v_1] \quad (5.45)$$

Since this amounts to a constant shift of the stress tensor operator, it will have no effect on the noise kernel or fluctuations, as they are the variance about the mean. This is further supported by the fact that there is no noise associated with the trace. Since this result was derived by only assuming that the Green function satisfies the field equation in *one* of its variables, it is independent of the issue of the lack of symmetry in the Hadamard ansatz (5.40).

Using the above formalism we now derive the coincident limit expression for the noise kernel (5.31). To get a meaningful result, we must work with the regularization of the Wightman function, obtained by following the same procedure outlined above for the Hadamard function:

$$G_{\text{ren}}(x, y) \equiv G_{\text{ren},+}(x, y) = G_+(x, y) - G^L(x, y) \quad (5.46)$$

In doing this, we assume the singular structure of the Wightman function is the same as that for the Hadamard function. In all applications, this is indeed the case. Moreover, when we compute the coincident limit of  $N_{abc'd'}$ , we will be working in the Euclidean section where there is no issue of operator ordering. For now we only consider spacetimes with no time dependence present in the final coincident limit result, so there is also no issue of Wick rotation back to a Minkowski signature. If this was the case, then care must be taken as to whether we are considering  $[N_{abc'd'}[G_{\text{ren},+}(x, y)]]$  or  $[N_{abc'd'}[G_{\text{ren},+}(y, x)]]$ .

We now have all the information we need to compute the coincident limit of the noise kernel (5.31). Since the point separated noise kernel  $N_{abc'd'}(x, y)$  involves covariant derivatives at the two points at which it has support, when we take the coincident limit we can use Synge's theorem (5.3) to move the derivatives acting at  $y$  to ones acting at  $x$ . Due to the long length of the noise kernel expression, we will only give an example by examining a single term.

Consider a typical term from the noise kernel functional (5.31):

$$G_{\text{ren};c'b} G_{\text{ren};d'a} + G_{\text{ren};c'a} G_{\text{ren};d'b} \quad (5.47)$$

Recall the noise kernel itself is related to the noise kernel functional via

$$N_{abc'd'} = N_{abc'd'}[G_{\text{ren}}(x, y)] + N_{abc'd'}[G_{\text{ren}}(y, x)]. \quad (5.48)$$

This is implemented on our typical term by adding to it the same term, but now with the roles of  $x$  and  $y$  reversed, so we have to consider

$$G_{\text{ren};c'b} G_{\text{ren};d'a} + G_{\text{ren};a'd} G_{\text{ren};b'c} + G_{\text{ren};c'a} G_{\text{ren};d'b} + G_{\text{ren};a'c} G_{\text{ren};b'd} \quad (5.49)$$

It is to this form that we can take the coincident limit:

$$[G_{\text{ren};c'b}] [G_{\text{ren};d'a}] + [G_{\text{ren};a'd}] [G_{\text{ren};b'c}] + [G_{\text{ren};c'a}] [G_{\text{ren};d'b}] + [G_{\text{ren};a'c}] [G_{\text{ren};b'd}] \quad (5.50)$$

We can now apply Sygne's theorem:

$$\begin{aligned} & ([G_{\text{ren};a}]_d - [G_{\text{ren};ad}]) ([G_{\text{ren};b}]_c - [G_{\text{ren};bc}]) \\ & + ([G_{\text{ren};d}]_a - [G_{\text{ren};ad}]) ([G_{\text{ren};c}]_b - [G_{\text{ren};bc}]) \\ & + ([G_{\text{ren};a}]_c - [G_{\text{ren};ac}]) ([G_{\text{ren};b}]_d - [G_{\text{ren};bd}]) \\ & + ([G_{\text{ren};c}]_a - [G_{\text{ren};ac}]) ([G_{\text{ren};d}]_b - [G_{\text{ren};bd}]) . \end{aligned} \quad (5.51)$$

This is the desired form for once we have an end point expansion of  $G_{\text{ren}}$ , it will be straightforward to compute the above expression. The details of such an evaluation in the context of symbolic computations can be found in [115].

The final result for the coincident limit of the noise kernel is broken down into a rank four and rank two tensor and a scalar according to

$$[N_{abc'd'}] = N'_{abcd} + g_{ab}N''_{cd} + g_{cd}N''_{ab} + g_{ab}g_{cd}N'''. \quad (5.52a)$$

The complete expression is given in Ref. [9].

## D. Summary Statements

### 1. Further Developments

In this section we showed how to obtain a general expression for the noise kernel, or the vacuum expectation value of the stress energy bi-tensor for a quantum scalar field in a general curved space time using the point separation method. The general form is expressed as products of covariant derivatives of the quantum field's Green function. It is finite when the noise kernel is evaluated for distinct pairs of points (and non-null points for a massless field). We also have shown the trace of the noise kernel vanishes for massless conformal fields, confirming there is no noise associated with the trace anomaly. This holds regardless of issues of regularization of the noise kernel.

The noise kernel as a two point function of the stress energy tensor diverges as the pair of points are brought together, representing the “standard” ultraviolet divergence present in the (point-defined) quantum field theory. By using the modified point separation regularization method we render the field's Green function finite in the coincident limit. This in turn permits the derivation of the formal expression for the regularized coincident limit of the noise kernel.

The general results obtained here are now applied by Phillips and Hu to compute the regularized noise kernel for three different groups of spacetimes: 1) ultrastatic metrics including the Einstein universe [62], hot flat space and optical Schwarzschild spacetimes. 2) Robertson-Walker universe and Schwarzschild black holes, from which structure formation from quantum fluctuations [40] and backreaction of Hawking radiation on the black hole spacetime [47] can be studied. 3) de Sitter and anti-de Sitter spacetimes: The former is necessary for scrutinizing primordial fluctuations in the cosmic background radiation while the latter is related to black hole phase transition and AdS/CFT issues in string theory.

When the Green function is available in closed analytic form, as is the case for optical metrics including hot flat space and Einstein Universe, conformally static spaces such as Robertson-Walker Unviwerses, and maximally symmetric spaces such as de Sitter and Anti de Sitter , one can carry out an end point expansion according to (5.5), displaying the ultraviolet divergence. Subtraction of the Hadamard ansatz (5.40), expressed as a series expansion (5.41), will render this Green function finite in the coincident limit. With this, one can calculate the noise kernel for a variety of spacetimes.

### 2. The Meaning of Regularization Revisited

Thus far we have focussed on isolating and removing the divergences in the stress energy bitensor and the noise kernel in the coincident limit. In closing, we would like to redirect the reader's attention to the intrinsic values of the point-separated geometric quantities like the bitensor. As stressed earlier [3], in the stochastic gravity program,

point separated expression of stress energy bi-tensor have fundamental physical meaning as it contains information on the fluctuations and correlation of quantum fields, and by consistency with the gravity sector, can provide a probe into the coherent properties of quantum spacetimes. Taking this view, we may also gain a new perspective on ordinary quantum field theory defined on single points: The coincidence limit depicts the low energy limit of the full quantum theory of matter and spacetimes. Ordinary (pointwise) quantum field theory, classical general relativity and semiclassical gravity are the lowest levels of approximations and should be viewed not as fundamental, but only as effective theories. As such, even the way how the conventional point-defined field theory emerges from the full theory when the two points (e.g.,  $x$  and  $y$  in the noise kernel) are brought together is interesting. For example, one can ask if there will also be a quantum to classical transition in spacetime accompanying the coincident limit? Certain aspects like decoherence has been investigated before (see, e.g., [27]), but here the non-local structure of spacetime and their impact on quantum field theory become the central issue. (This may also be a relevant issue in noncommutative geometry). The point-wise limit of field theory of course has ultraviolet divergences and requires regularization. A new viewpoint towards regularization evolved from this perspective of treating conventional pointwise field theory as an effective theory in the coincident limit of the point-separated theory of extended spacetime.

To end this discussion, we venture one philosophical point we find resounding in these investigations reported here. It has to do with the meaning of a point-defined versus a point-separated field theory, the former we take as an effective theory coarse-grained from the latter, the point-separated theory reflecting a finer level of spacetime structure. It bears on the meaning of regularization, not just at the level of technical procedures, but related to finding an effective description and matching with physics observed at a coarser scale or lower energy.

In particular, we feel that finding a finite energy momentum tensor (and its fluctuations as we do here) which occupied the center of attention in the research of quantum field theory in curved spacetime in the 70's is only a small part of a much larger and richer structure of theories of fields and spacetimes. We come to understand that whatever regularization method one uses to get these finite parts in a point-wise field theory should not be viewed as universally imparting meaning beyond its specified function, i.e., to identify the divergent pieces and provide a prescription for their removal. We believe the extended structure of spacetime (e.g., via point-separation or smearing) and the field theory defined therein has its own much fuller meaning beyond just reproducing the well-recognized result in ordinary quantum field theory as we take the point-wise or coincident limit. In this way of thinking, the divergence-causing terms are only 'bad' when they are forced to a point-wise limit, because of our present inability to observe or resolve otherwise. If we accord them with the full right of existence beyond this limit, and acknowledge that their misbehavior is really due to our own inability to cope, we will be rewarded with the discovery of new physical phenomena and ideas of a more intricate world. (Maybe this is just another way to appreciate the already well-heeded paths of string theory.)

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FIG. 1. The dimensionless fluctuation measure  $\Delta \equiv (\langle \hat{\rho}^2 \rangle - \langle \hat{\rho} \rangle^2) / \langle \hat{\rho}^2 \rangle$  for the Casimir topology, along with  $\Delta_{L,\text{Reg}}$ . The topology is that of a flat three spatial dimension manifold with one periodic dimension of period  $L = 1$ . The smearing width  $\sigma$  represents the sampling width of the energy density operator  $\hat{\rho}(\sigma)$ .  $\Delta$  is for the complete fluctuations, including divergent and cross terms, while  $\Delta_{L,\text{Reg}}$  is just for the finite parts of the mean energy density and fluctuations.

FIG. 2. The finite parts of the mean energy density  $\rho^{\text{fin}}(\sigma, L)$  and the fluctuations  $\Delta\rho^{\text{fin}}(\sigma, L)$  for the Casimir topology, as a function of the smearing width.

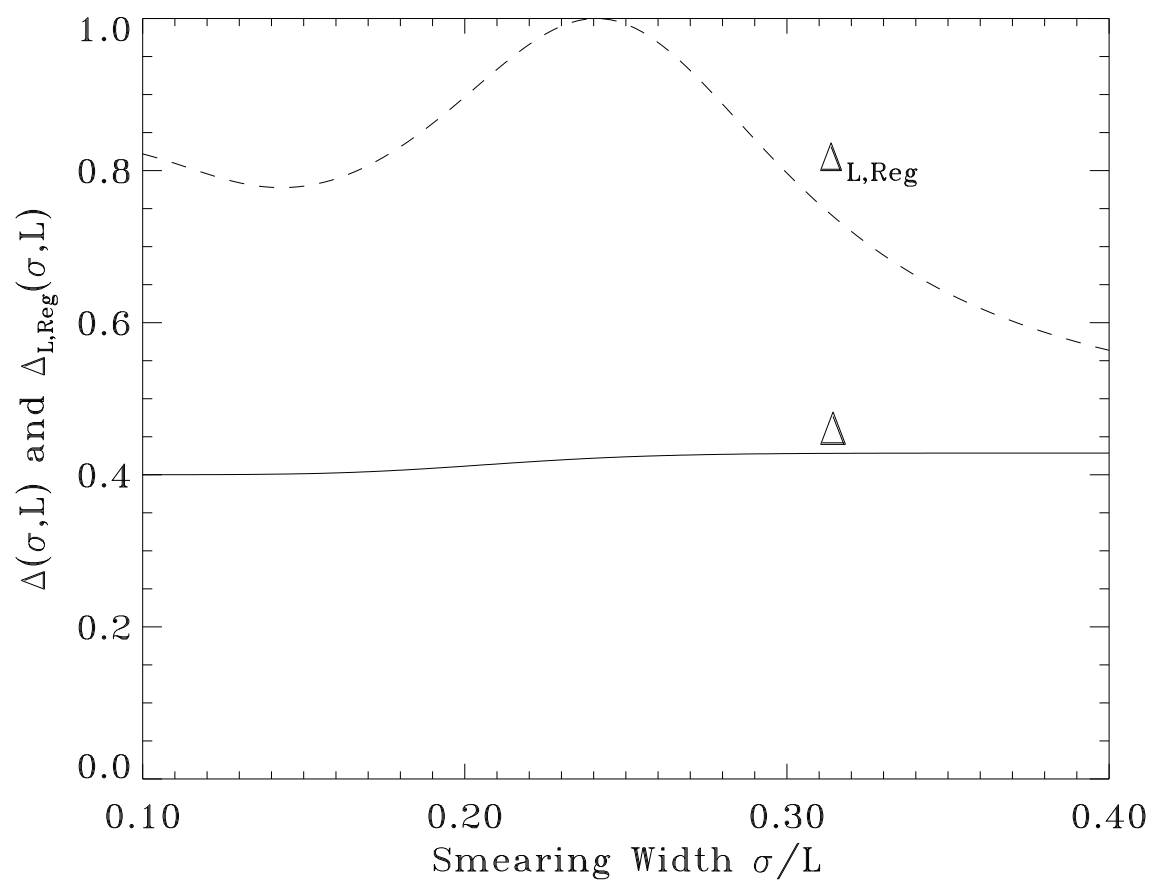


Fig 1, N.G.Phillips

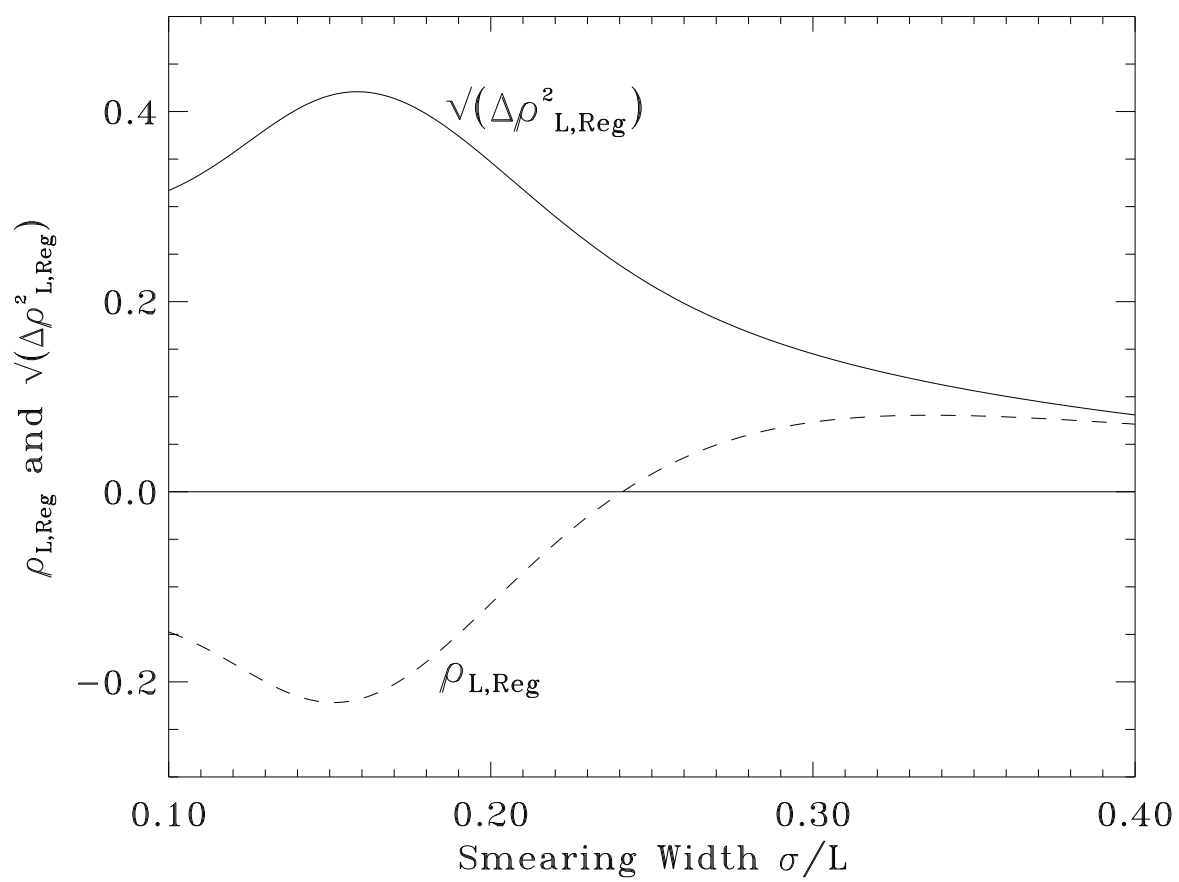


Fig 2, N.G.Phillips